Parametric Inference and Dynamic State Recovery from Option Panels

Torben G. Andersen

Joint work with Nicola Fusari and Viktor Todorov

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Motivation

• Under realistic assumptions: Derivatives Non-Redundant Assets.

• Contain important Information about Volatility and Jump risks and their Pricing:

  \[ \text{Option Price} \iff \text{Volatility} \& \text{Jumps} + \text{Time-Varying Risk Premia} + \text{Observation Error} \]

• Derivatives on Equity Indices actively Traded:
  – On Average, 200 plus SPX Option Quotes at Close of Trading,
  – Cover a wide range of Moneyness and Tenor (time-to-maturity).
Motivation

Most Parametric Option-based Estimation Methods of Risk Premia follow Two Steps:

1. Identify Volatility and Jump Risks from underlying Asset Data,

2. Use Price Levels in Option Panel to estimate Risk Premia:
   - typically via restrictive Specification; i.e., small $\mathbb{P} - \mathbb{Q}$ wedge.
   - Option Price Error ignored or modeled with Normal Distribution.
IV error kernel regression

Log Moneyness: \( \log(\frac{K}{F}) \)

$$\sigma \sqrt{\tau}$$

IV_a - IV_b

\( \frac{IV_a + IV_b}{2} \)
**Motivation**

Goal to develop Estimation Technique that:

1. fully uses the **State Dynamics** implied by Option Prices;
2. is Robust with respect to **Option Price Error Specification**;
3. relies on **In-Fill Asymptotics** (Increasing number of Options each Trading Day);
4. Specifies only **Risk-Neutral Dynamics** (allows for flexible risk premia).

**In Sum**: Formal **Estimation, Inference and Diagnostic Tests** for Option Pricing

- Obtain **Path of State Vector Realizations** solely from Option Panel
- Set Stage for **Risk Premia Estimation** via Semi-Parametric $\mathbb{P}$ Estimation
Underlying price process

Option cross section

Maturity

Moneyness

$t_1$

$t_2$

$t_3$
Outline

• Information in Option Panels

• Inference in Presence of Noise

• Semiparametric Tests
Notation

Formally, underlying Price $X_t$ has the following $\mathbb{P}$-Dynamics:

$$\frac{dX_t}{X_t} = \alpha_t \, dt + \sqrt{V_t} \, dW_t^\mathbb{P} + \int_{x > -1} x \tilde{\mu}^\mathbb{P}(dt, dx),$$

$W^\mathbb{P}_t$ is a Brownian motion; $\sqrt{V_t}$ is \textit{Spot} Volatility (under both $\mathbb{P}$ and $\mathbb{Q}$);

$\mu^\mathbb{P}$ is an Integer-valued Random Measure; $\tilde{\mu}^\mathbb{P} = \mu^\mathbb{P} - \nu^\mathbb{P}$

$\mu^\mathbb{P}$ Counts Jumps in $X$; Jump Compensator is $\nu^\mathbb{P}(ds, dx)$.

We assume $V_t = \xi_1(S_t)$, where $S_t$ is Latent State Vector ($p \times 1$).
**Notation**

**Assumption A0.** The process $X$, defined over the fixed interval $[0, T]$, satisfies:

1. For $s, t \geq 0$, exists $K > 0$: $\mathbb{E} \{ |V_t - V_s|^2 \wedge K \} \leq K|t - s|$.

2. $\int_{x > -1} (|x|^\beta \wedge 1) \nu^p(dx) < \infty$, for some $\beta \in [0, 2)$.

3. $\inf_{t \in [0,T]} V_t > 0$ and the processes $\alpha_t$, $V_t$ and $a_t$ are locally bounded.

A0(i) satisfied if $V_t$ is governed by (multivariate) Stochastic Differential Equation

A0(ii) restricts so-called Blumenthal-Getoor index of the jumps to be below $\beta$

A0(iii) implies, at each $t \in [0, T]$, the price process has Non-Vanishing BM Component

Assumption A0 **does not involve Integrability or Stationarity Conditions** for the Model
Likewise, $X_t$ has $\mathbb{Q}$-Dynamics:

$$\frac{dX_t}{X_{t^-}} = (r_t - \delta_t) dt + \sqrt{V_t} dW_t + \int_{x > -1} x \tilde{\mu}(dt, dx),$$

$$\nu(dt, dx) = \xi_2(S_t) \otimes \nu(dx), \quad \text{where} \quad \tilde{\mu} = \mu - \nu$$

We denote Options with Log-Moneyness $k = \log(K/X_t)$ and Tenor $\tau$ by

$$O_{t,k,\tau} = \mathbb{E}_{t}^{\mathbb{Q}} \left\{ e^{-\int_{t}^{t+\tau} (r_s - \delta_s) ds} (X_{t+\tau} - K)^+ \right\},$$

We denote associated Black-Scholes Implied Volatility by $\kappa(k, \tau, S_t)$. 
An Empirical Illustration

The “Double-Jump” Model of Duffie, Pan and Singleton (2001) has Risk-Neutral Dynamics:

\[
\frac{dX_t}{X_t} = (r - \delta) dt + \sqrt{V_t} dW_t + dL_{x,t},
\]

\[
dV_t = \kappa (\bar{V} - V_t) dt + \sigma_d \sqrt{V_t} dB_t + dL_{v,t},
\]

$L_{x,t}$ and $L_{v,t}$ are Jump Martingales;

$(L_{x,t}, L_{v,t})$ Jump (simultaneously) with i.i.d. Probability $\lambda_j$, Jump Size $(Z_x, Z_v)$.

$Z_v \sim \text{exp} (\mu_v)$; $\log (Z_x + 1) | Z_v \sim \mathcal{N} (\mu_x + \rho_j Z_v, \sigma^2_x)$; $\text{Cor} (dW_t, dB_t) = \rho_d$

State Vector (Realization):

$$\{V_t\}_{t=1}^{T}$$

Risk-Neutral Parameters:

$$\theta = (\rho_d, \bar{V}, \kappa, \sigma_d, \lambda, \mu_x, \sigma_x, \mu_v, \rho_j)'.$$
Monte Carlo Scenario

Inspired by Calibration/Empirical Estimates from BCJ (2007); $\rho_j = 0$.

Table 1: Parameter Setting for the Numerical Experiments

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_d$</td>
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<td>$\lambda_j$</td>
<td>1.0080</td>
<td>$\rho_d$</td>
<td>-0.4600</td>
<td>$\lambda_j$</td>
<td>1.0080</td>
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<td>$\nu$</td>
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<td>$\mu_x$</td>
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<td>$\nu$</td>
<td>0.0144</td>
<td>$\mu_x$</td>
<td>-0.0501</td>
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<td>$\sigma_x$</td>
<td>0.0490</td>
<td>$\kappa_d$</td>
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<td>$\sigma_x$</td>
<td>0.0751</td>
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<tr>
<td>$\sigma_d$</td>
<td>0.2000</td>
<td>$\mu_v$</td>
<td>0.0315</td>
<td>$\sigma_d$</td>
<td>0.2000</td>
<td>$\mu_v$</td>
<td>0.0930</td>
</tr>
</tbody>
</table>

Observation Errors on Option Prices: $\epsilon_{t,k,\tau} = \sigma_{t,k,\tau} Z_{t,k,\tau}$, $Z_{t,k,\tau} \sim \mathcal{N}(0, 1)$.

$\sigma_{t,k,\tau} = \frac{1}{2} \psi_k / Q_{0.995}$, $\psi_k =$ Bid-Ask Spread “at $k$,” $Q_p = p^{th}$ Quantile of $\mathcal{N}(0, 1)$.
Information in Panels of Options

What can we Identify from Options?

- Different Parts of Volatility Surface Load differently on distinct Risks and their Pricing:
  - Short-Term OTM Options determined largely by Pricing of Jump Risks
  - Role of Volatility Risks more prominent for ATM Options
  - Different Maturities separate Persistent from Transient State Variables
  - Persistence of “Smirk” Identifies Sources of “Leverage” type Effects
  - ....

Can we identify the model?
Option Sensitivity to Parameters; Double-Jump Model
Information in Panels of Options

• A Large Cross-Section of Option Prices observed without Error can Identify Risk-Neutral Parameters and the Current Value of the State Vector,

• Once Risk-Neutral Parameters are Known, Options are Known Transformations of the State Variables → Contain Same Information as observing directly the State Vector,

• Options alone Contain Information to Estimate the Risk Premia!
Assumption A1. Fix $T > 0$. For each Date $t = 1, \ldots, T$ and Moneyness $\tau$, \# options $N^\tau_t \uparrow \infty$ with $N^\tau_t / N_t \to \pi^\tau_t$ and $N_t / \sum_{t=1}^T N_t \to \varsigma_t$, where $\pi^\tau_t, \varsigma_t > 0$.

Let $\underline{k}(t, \tau), \overline{k}(t, \tau)$ denote Min, Max Log-Moneyness on Day $t$, Maturity $\tau$.

Sequence of Grid Nested: $\underline{k}(t, \tau) = k_{t,\tau}(0) < k_{t,\tau}(1) < \ldots < k_{t,\tau}(N^\tau_t) = \overline{k}(t, \tau)$.

$N_t \cdot (k_{t,\tau}(i) - k_{t,\tau}(i - 1)) \to \psi_{t,\tau}(k)$ Uniformly on $(\underline{k}(t, \tau), \overline{k}(t, \tau))$.

Assumption A2. For every $\epsilon > 0$ and $T > 0$ finite, we have a.s.

$$\inf_{t=1, \ldots, T: \cup \|Z_t - S_t\| > \epsilon \cup \|\theta - \theta_0\| > \epsilon} \sum_{t=1}^T \sum_{\tau} \int_{\underline{k}(t, \tau)}^{\overline{k}(t, \tau)} (\kappa(k, \tau, S_t, \theta_0) - \kappa(k, \tau, Z_t, \theta))^2 dk > 0,$$

where $\theta$ is the risk-neutral parameter vector.
Inference in the Presence of Noise

Options are Observed with Error, i.e., we observe $\hat{\kappa}_{t,k,\tau}$ for

$$\hat{\kappa}_{t,k,\tau} = \kappa_{t,k,\tau} + \epsilon_{t,k,\tau},$$

where the errors, $\epsilon_{t,k,\tau}$, are defined on an extension of the original probability space.

We assume the Error can be averaged out by Pooling Options across Moneyness:

**Assumption A3.** For every $\epsilon > 0$ and $T > 0$ finite, we have

$$\sup_{t=1,\ldots,T} \sum_{j=1}^{N_t} \frac{1}{N_t} \left( \kappa(k_j, \tau_j, S_t, \theta_0) - \kappa(k_j, \tau_j, Z_t, \theta) \right) \epsilon_{t,k,\tau} \rightarrow 0, \quad \mathbb{P},$$

when $\min_{t=1,\ldots,T} N_t \rightarrow \infty$ for all $\theta \in \Theta$. 
We define our estimator of risk-neutral parameters and state variables as

$$
(\{S^n_t\}_{t=1,...,T}, \hat{\theta}^n) = \arg\min_{\{z_t\}_{t=1,...,T}, \theta \in \Theta} \sum_{t=1}^T \left\{ \frac{1}{N_t} \sum_{j=1}^{N_t} (\hat{\kappa}^n_{t,k,\tau} - \kappa(k_j, \tau_j, z_t, \theta))^2 + \lambda_n (\hat{V}^n_t - \xi_1(z_t))^2 \right\},
$$

$$\lambda_n \geq 0, \quad \hat{V}^n_t \text{ is Nonparametric Estimator of Volatility from High-Frequency Data.}$$
Underlying price process

Option cross section

Maturity

Moneyness

$\text{t}_3$

$\text{t}_2$

$\text{t}_1$
Estimation

**Theorem 1.** Suppose Assumptions A1-A3 Hold for some $T \in \mathbb{N}$ fixed, and \( \{ \hat{V}_t^n \}_{t=1,\ldots,T} \) is Consistent for \( \{ V_t \}_{t=1,\ldots,T} \), as \( n \to \infty \).

Then, if \( \min_{t=1,\ldots,T} N_t \to \infty \) and \( \lambda_n \to \lambda \) for some finite \( \lambda \geq 0 \), as \( n \to \infty \), we have that \( (\hat{S}_t^n, \hat{\theta}_t^n) \) exists with probability approaching 1, and

\[
\| \hat{S}_t^n - S_t \| \xrightarrow{\mathbb{P}} 0, \quad \| \hat{\theta}_t^n - \theta_0 \| \xrightarrow{\mathbb{P}} 0, \quad t = 1, \ldots, T.
\]
Estimation

To quantify precision of estimation we need slightly stronger assumption on errors:

**Assumption A4.** *For the error process, $\epsilon_{t,k,\tau}$, we have,

(i) $\mathbb{E}\left(\epsilon_{t,k,\tau} | \mathcal{F}^{(0)}\right) = 0$

(ii) $\mathbb{E}\left(\epsilon_{t,k,\tau}^2 | \mathcal{F}^{(0)}\right) = \phi_{t,k,\tau}$, for $\phi_{t,k,\tau}$ continuous in its second argument

(iii) $\epsilon_{t,k,\tau}$, $\epsilon_{t',k',\tau'}$ are independent, conditional on $\mathcal{F}^{(0)}$, for $(t, k, \tau) \neq (t', k', \tau')$

(iv) $\mathbb{E}\left( |\epsilon_{t,k,\tau}|^4 | \mathcal{F}^{(0)}\right) < \infty$, almost surely

where $\mathcal{F}^{(0)}$ is the $\sigma$-algebra associated with $X$. 
**Estimation**

**Theorem 2.** Assume Assumptions A1-A4 Satisfied for $T \in \mathbb{N}$ Fixed, and $\kappa(t, \tau, Z, \theta)$ is twice Continuously-Differentiable in its arguments.

Then, if $\min_{t=1, \ldots, T} N_t \to \infty$ and $\lambda_n^2 \min_{t=1, \ldots, T} N_t \to 0$, for $n \to \infty$:

$$
\begin{pmatrix}
\sqrt{N_1} (\hat{S}_1^n - S_1) \\
\vdots \\
\sqrt{N_T} (\hat{S}_T^n - S_T) \\
\sqrt{\frac{N_1 + \ldots + N_T}{T}} (\hat{\theta}^n - \theta_0)
\end{pmatrix} \xrightarrow{L^2} \mathbf{H}_T^{-1} \left( \Omega_T \right)^{1/2} \begin{pmatrix} E_1 \\ \vdots \\ E_T \end{pmatrix} + \begin{pmatrix} E'_1 \\ \vdots \\ E'_{T-1} \end{pmatrix} \text{,}
$$

$E_1, \ldots, E_T$ are $p \times 1$ vectors, $E'$ is $q \times 1$ vector, all are i.i.d. Standard Normal and Defined on an Extension of the original Probability Space,

$\mathbf{H}_T, \Omega_T$ are $\mathcal{F}_T^{(0)}$-adapted Random Matrices, for which Consistent Estimates can be Constructed from Options Data.
MC Estimation; Double-Jump Model; 1,000 Replications
Empirical Application

We use the following Data Set in the Application

- CBOE European-style (SPX) Options on the S&P 500 index,
- The Options have Maturity Ranging from 8 Days to 1 Year,
- The Data Covers Period 1996 — 2010 for a Total of 3,500 Days,
- We Apply Standard Filters; Retain only OTM and ATM Options; Wide Strike Range,
- For Semi-Parametric Tests: 5-minute S&P 500 Futures, Same Sample Period.
Model-free vs Option-Implied Volatility

- We specify and estimate the risk-neutral distribution of the underlying process $X$ and we do not impose any parametric structure for the dynamics under the true statistical measure $\mathbb{P}$.

- Absence of arbitrage implies that recovered volatility from options should be the same with that “observed” in the underlying asset $X$.

- This is a semiparametric restriction: it is based on a parametric specification for the risk-neutral distribution as well as nonparametric estimate for the stochastic volatility.
An Empirical Illustration

The “Double-Jump” Model of Duffie, Pan and Singleton (2001) has Risk-Neutral Dynamics:

\[ \frac{dX_t}{X_{t-}} = (r - \delta) dt + \sqrt{V_t} dW_t + dL_{x,t}, \]

\[ dV_t = \kappa (\bar{v} - V_t) dt + \sigma_d \sqrt{V_t} dB_t + dL_{v,t}, \]

\( L_{x,t} \) and \( L_{v,t} \) are Jump Martingales;

\( (L_{x,t}, L_{v,t}) \) Jump (simultaneously) with i.i.d. Probability \( \lambda_j \), Jump Size \( (Z_x, Z_v) \).

\( Z_v \sim \exp (\mu_v); \log (Z_x + 1) \vert Z_v \sim \mathcal{N} (\mu_x + \rho_j Z_v, \sigma_x^2); \ Cor (dW_t, dB_t) = \rho_d \)

**State Vector (Realization):** \( \{V_t\}_{t=1}^T \)

**Risk-Neutral Parameters:** \( \theta = (\rho_d, \bar{v}, \kappa, \sigma_d, \lambda, \mu_x, \sigma_x, \mu_v, \rho_j)' \).
Empirical Application

Qualitative Features of the Double-Jump Model

- Both Stochastic Volatility and Price Jumps Present,
- Volatility can Move through Jumps,
- Price and Volatility may be Correlated through Small and Big Moves,
- Jump Intensity is Constant.
- (Only) One Volatility Factor.
- Slightly more General than BCJ “State-of-Art”
## Empirical Application

### Table 2: Parameter Estimates of One-Factor Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_d$</td>
<td>-0.9586</td>
<td>0.0059</td>
<td>$\lambda$</td>
<td>0.0150</td>
<td>0.0006</td>
</tr>
<tr>
<td>$\overline{v}$</td>
<td>0.0356</td>
<td>0.0004</td>
<td>$\mu_x$</td>
<td>0.5833</td>
<td>0.0417</td>
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<tr>
<td>$\kappa$</td>
<td>1.4350</td>
<td>0.0198</td>
<td>$\sigma_x$</td>
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<td>0.0430</td>
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<td>$\sigma_d$</td>
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<td>0.0032</td>
<td>$\mu_v$</td>
<td>1.6162</td>
<td>0.1912</td>
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<tr>
<td>$\rho_j$</td>
<td>-0.8579</td>
<td>0.1572</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Risk Neutral Mean of Volatility: 24.3% vs Sample RV Estimate: 21.35%

Mean Risk-Neutral (Log-Return) Jump: −80% Annual Jump Probability: 1.5%

Mean Risk-Neutral Volatility (Level) Jump: $\sqrt{1.62} = 127\%$
Diagnostic Tests

We design the following Diagnostic Tests of Model Performance:

- Fit to the Volatility Surface over some Period of Time

- Parameter Stability across Time

- Distance between Model-Free and Option-Model Implied Volatility
Diagnostic Test I: Fit to Volatility Surface

Corollary 1. Let $\mathcal{K} \subset \left( \underline{k}(t, \tau^*), \overline{k}(t, \tau^*) \right)$ be a set with positive Lebesgue measure and $N_t^K$ be the number of options on day $t$ with tenor $\tau^*$ and log-moneyness in $\mathcal{K}$. Then, given our Assumptions, we have,

$$
\sum_{j: k_j \in \mathcal{K}} \left( \frac{\hat{\kappa}_{t,k_j,\tau^*} - \kappa(k_j, \tau^*, \hat{S}_t^n, \hat{\theta}^n)}{\sqrt{\hat{\Pi}_T' \hat{\Xi}_T \hat{\Pi}_T}} \right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1),
$$

where $\hat{\Pi}_T$ and $\hat{\Xi}_T$ are some $\mathcal{F}_T^{(0)}$-adapted random matrices.
Z-score: short maturity DOTM Put options

Z-score: short maturity OTM Put options

Z-score: short maturity OTM Call options
Z-score: long maturity DOTM Put options

Z-score: long maturity OTM Put options

Z-score: long maturity OTM Call options
Diagnostic Test II: Parameter Stability

Parameters Estimated over Non-Overlapping Periods should, up to Statistical Error, be Identical. Thus,

\[
\left(\hat{\theta}_1^n - \hat{\theta}_2^n\right)' \left(\hat{\text{Avar}}(\hat{\theta}_1^n) + \hat{\text{Avar}}(\hat{\theta}_2^n)\right)^{-1} \left(\hat{\theta}_1^n - \hat{\theta}_2^n\right) \xrightarrow{L^s} \chi^2(q),
\]

where \(\hat{\text{Avar}}(\hat{\theta}_j^n)\) is Consistent Asymptotic Variance Estimate for \(\hat{\theta}_j^n\).

Note:

- Under Model Misspecification, Parameter Estimates Converge to Pseudo-True Values.
- However, as State Vector changes over Time \(\Rightarrow\) Pseudo-True Values Change as well.
# SPX Options – Parameter Stability Test

## Table 3: Parameter Stability; S&P 500 Options Data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Nominal size of test</th>
<th>Parameter</th>
<th>Nominal size of test</th>
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<tbody>
<tr>
<td></td>
<td>1%</td>
<td>5%</td>
<td>1%</td>
</tr>
<tr>
<td>Panel A: One-Factor Model</td>
<td></td>
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<td></td>
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<tr>
<td>$\rho_d$</td>
<td>62.86%</td>
<td>70.48%</td>
<td>$\lambda_j$</td>
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<tr>
<td>$\bar{\nu}$</td>
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<td>73.33%</td>
<td>$\mu_x$</td>
</tr>
<tr>
<td>$\kappa$</td>
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<td>93.33%</td>
<td>$\sigma_x$</td>
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<tr>
<td>$\sigma_d$</td>
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<tr>
<td>$\rho_j$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Panel B: Two-Factor Model</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho_{d,1}$</td>
<td>8.57%</td>
<td>16.19%</td>
<td>$\lambda_{j,0}$</td>
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<tr>
<td>$\bar{\nu}_1$</td>
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<td>12.38%</td>
<td>$\lambda_{j,1}$</td>
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<tr>
<td>$\kappa_{d,1}$</td>
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<td>70.05%</td>
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</tr>
<tr>
<td>$\sigma_{d,1}$</td>
<td>42.86%</td>
<td>53.33%</td>
<td>$\sigma_x$</td>
</tr>
<tr>
<td>$\rho_{d,2}$</td>
<td>7.62%</td>
<td>16.19%</td>
<td>$\mu_v$</td>
</tr>
<tr>
<td>$\bar{\nu}_2$</td>
<td>76.19%</td>
<td>80.95%</td>
<td>$\rho_j$</td>
</tr>
</tbody>
</table>
Model-Free vs Option-Implied Volatility

Two Nonparametric Estimators for Spot Volatility from High-Frequency Data:

\[ \hat{V}_{t}^{\pm,n} = \frac{n}{k_n} \sum_{i \in I^{\pm,n}} (\Delta_{i}^{t,n} X)^2 1 \left( |\Delta_{i}^{t,n} X| \leq \alpha n^{-\varpi} \right), \quad \Delta_{i}^{t,n} X = \log \left( X_{t+i/n} \right) - \log \left( X_{t+(i-1)/n} \right), \]

where \( \alpha > 0, \ \varpi \in (0, 1/2), \ k_n \) is Deterministic sequence, \( k_n/n \to 0 \), and

\[ I^{-,n} = \{-k_n + 1, \ldots, 0\} \quad \text{and} \quad I^{+,n} = \{1, \ldots, k_n\}. \]

- \( V_{t}^{-,n} \) is Estimator for Spot Variance from Left; \( V_{t}^{+,n} \) Estimator from Right.
- \( V_{t}^{-,n} \) and \( V_{t}^{+,n} \) Differ only if Volatility Jumps at \( t \) (Probability Zero Event).
Corollary 2. Under the same conditions as in Theorem 3, we have for $k_n \to \infty$, $\min_{t=1,...,T} N_t \to \infty$ and $\lambda_n^2 \min_{t=1,...,T} N_t \to 0$,

$$\left\{ \begin{array}{c} \xi_1(\hat{S}_t^n) - \hat{V}_t^{+,n} \\ \sqrt{\nabla S_1(\hat{S}_t^n)'\hat{\chi}_t \nabla S_1(\hat{S}_t^n) / N_t} + 2(\hat{V}_t^{+,n})^2 / kn \end{array} \right\}_{t=1,...,T} \overset{\mathcal{L}}{\to} \left( \begin{array}{c} \hat{E}_1 \\ \vdots \\ \hat{E}_T \end{array} \right),$$

where $\hat{\chi}_t$ is the part of $\hat{H}_T^{-1}\hat{\Omega}_T(\hat{H}_T^{-1})'$ corresponding to the variance-covariance of $\hat{S}_t^n$ and $(\hat{E}_1, ..., \hat{E}_T)'$ is a vector of standard normals independent of each other and of $\mathcal{F}$. 

Diagnostic Test III: Model-free vs Option-Implied Volatility
Nonparametric volatility estimate

Option recovered volatility

Z-score: recovered − nonparametric volatility

ACF: in level

ACF: in log
Empirical Application – Two-Factor Model

We now extend the Model to Include Two SV Factors:

\[
\frac{dX_t}{X_{t-}} = (r - \delta)dt + \sqrt{V_{1,t}}dW_{1,t} + \sqrt{V_{2,t}}dW_{2,t} + dL_{x,t},
\]

\[
dV_{1,t} = \kappa_1(\bar{v}_1 - V_{1,t})dt + \sigma_{1,d}\sqrt{V_{1,t}}dB_{1,t} + dL_{v,t},
\]

\[
dV_{2,t} = \kappa_2(\bar{v}_2 - V_{2,t})dt + \sigma_{2,d}\sqrt{V_{2,t}}dB_{2,t},
\]

Now, Jump Intensity is \(\lambda_0 + \lambda_1 V_{1,t}\): Jumps Self-Exciting, as Jumps Impact Volatility.

Note:

- Extension allows “Fears” to be Time-Varying.
- Breaks Tight Link between Pricing of Risk and its Level.
## Empirical Application

Table 4: Parameter Estimates of Two-Factor Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard Error</th>
<th>Parameter</th>
<th>Estimate</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_{1,d}$</td>
<td>-0.9800</td>
<td>0.0253</td>
<td>$\lambda_0$</td>
<td>0.0217</td>
<td>0.0027</td>
</tr>
<tr>
<td>$\bar{\nu}_1$</td>
<td>0.0331</td>
<td>0.0019</td>
<td>$\lambda_1$</td>
<td>6.0683</td>
<td>0.8749</td>
</tr>
<tr>
<td>$\kappa_1$</td>
<td>1.2327</td>
<td>0.0638</td>
<td>$\mu_x$</td>
<td>-0.0145</td>
<td>0.0136</td>
</tr>
<tr>
<td>$\sigma_{1,d}$</td>
<td>0.2640</td>
<td>0.0113</td>
<td>$\sigma_x$</td>
<td>0.0877</td>
<td>0.0082</td>
</tr>
<tr>
<td>$\rho_{2,d}$</td>
<td>-0.1824</td>
<td>0.0388</td>
<td>$\mu_v$</td>
<td>0.1501</td>
<td>0.0124</td>
</tr>
<tr>
<td>$\bar{\nu}_2$</td>
<td>0.0066</td>
<td>0.0001</td>
<td>$\rho_j$</td>
<td>-0.7756</td>
<td>0.0718</td>
</tr>
<tr>
<td>$\kappa_2$</td>
<td>29.8797</td>
<td>0.5951</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_{2,d}$</td>
<td>0.2341</td>
<td>0.0569</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Risk Neutral Mean of Volatility: 22.8\% vs Sample RV Estimate: 21.35\%


Mean Risk-Neutral Volatility (Level) Jump: $\sqrt{0.1501} = 38.7\%$
## SPX Options – Parameter Stability Test

Table 3: Parameter Stability; S&P 500 Options Data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Nominal size of test</th>
<th>Parameter</th>
<th>Nominal size of test</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1%</td>
<td>5%</td>
<td>1%</td>
</tr>
<tr>
<td>( \rho_d )</td>
<td>62.86%</td>
<td>70.48%</td>
<td>( \lambda_j )</td>
</tr>
<tr>
<td>( \bar{\nu} )</td>
<td>71.43%</td>
<td>73.33%</td>
<td>( \mu_x )</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>91.43%</td>
<td>93.33%</td>
<td>( \sigma_x )</td>
</tr>
<tr>
<td>( \sigma_d )</td>
<td>77.14%</td>
<td>80.95%</td>
<td>( \mu_v )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( \rho_j )</td>
</tr>
<tr>
<td>( \rho_{d,1} )</td>
<td>8.57%</td>
<td>16.19%</td>
<td>( \lambda_{j,0} )</td>
</tr>
<tr>
<td>( \bar{\nu}_1 )</td>
<td>7.62%</td>
<td>12.38%</td>
<td>( \lambda_{j,1} )</td>
</tr>
<tr>
<td>( \kappa_{d,1} )</td>
<td>72.38%</td>
<td>70.05%</td>
<td>( \mu_x )</td>
</tr>
<tr>
<td>( \sigma_{d,1} )</td>
<td>42.86%</td>
<td>53.33%</td>
<td>( \sigma_x )</td>
</tr>
<tr>
<td>( \rho_{d,2} )</td>
<td>7.62%</td>
<td>16.19%</td>
<td>( \mu_v )</td>
</tr>
<tr>
<td>( \bar{\nu}_2 )</td>
<td>76.19%</td>
<td>80.95%</td>
<td>( \rho_j )</td>
</tr>
<tr>
<td>( \kappa_{d,2} )</td>
<td>69.52%</td>
<td>77.14%</td>
<td></td>
</tr>
<tr>
<td>( \sigma_{d,2} )</td>
<td>0.95%</td>
<td>0.95%</td>
<td></td>
</tr>
</tbody>
</table>
## Diagnostic Tests I and III

### Table 5: Tests on S&P 500 options data

<table>
<thead>
<tr>
<th>Test</th>
<th>One-factor Model</th>
<th>Two-factor Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nominal size of test</td>
<td>Nominal size of test</td>
</tr>
<tr>
<td></td>
<td>1% 5%</td>
<td>1% 5%</td>
</tr>
<tr>
<td><strong>Panel A: Option Fit Tests</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DOTM, short-maturity puts</td>
<td>18.16% 39.74%</td>
<td>21.84% 45.53%</td>
</tr>
<tr>
<td>OTM, short-maturity puts</td>
<td>24.87% 53.03%</td>
<td>27.89% 50.39%</td>
</tr>
<tr>
<td>OTM, short-maturity calls</td>
<td>20.53% 55.00%</td>
<td>16.58% 48.55%</td>
</tr>
<tr>
<td>DOTM, long-maturity puts</td>
<td>41.45% 62.11%</td>
<td>27.50% 41.97%</td>
</tr>
<tr>
<td>OTM, long-maturity puts</td>
<td>72.63% 80.79%</td>
<td>52.89% 60.53%</td>
</tr>
<tr>
<td>OTM, long-maturity calls</td>
<td>53.16% 65.53%</td>
<td>79.47% 86.45%</td>
</tr>
<tr>
<td><strong>Panel B: Distance implied-nonparametric volatility</strong></td>
<td>54.08% 65.66%</td>
<td>49.74% 61.58%</td>
</tr>
<tr>
<td><strong>Panel C: Root-mean squared error of fit</strong></td>
<td>3.09%</td>
<td>2.32%</td>
</tr>
</tbody>
</table>
Empirical Application

Things to Note for Two-Factor Model:

- Fit Improves Significantly (RMSE Drops about 25%).
- Constant Part of Jump Intensity Small $\implies$ Jump Risk Premia Time-Varying.
- First Return-Volatility Correlation (remains) Extremely Negative.
- Second Volatility Factor Smaller, much Less Persistent.
- Model still Struggles with Short-Maturity OTM Calls, Long-Term OTM Options.
- Parameters Vary over time, particularly the ones driving Jump Distribution.
- In Quiet Period Jump Intensity Near Zero – Jump Parameters not Identified.
- Time-Varying Parameters $\implies$ Missing State Variables?
- Period 2006-2010 very Hard to Fit Reasonably.
- Model still Misspecified, even on Stretches of One Year.
Z-score: short maturity DOTM Put options

Z-score: short maturity OTM Put options

Z-score: short maturity OTM Call options
One-Factor Model – Fit to ATM Term Structure of IV

ATM IV short maturity

ATM IV long maturity

ATM IV short maturity – ATM IV long maturity
Two-Factor Model – Fit to ATM Term Structure of IV

ATM IV short maturity

ATM IV long maturity

ATM IV short maturity − ATM IV long maturity
Conclusions

• We Propose and Derive Asymptotic Properties of Estimation in Large Option Panels with Fixed Time Span and Increasing Cross-Section.

• Method requires Risk-Neutral Model only, is Nonparametric about Option Pricing Errors, and Allows for Heteroscedasticity in the latter.

• Battery of Statistics to Detect Sources of Model Misspecification:
  – Testing Model Fit Over Time and different Parts of Volatility Surface,
  – Testing Model Stability,
  – Testing Consistency between Model Option-Implicit Volatility and Nonparametric Estimate from High-Frequency Data on underlying Asset.