MINIMAL STATE VARIABLE SOLUTIONS TO MARKOV-SWITCHING RATIONAL EXPECTATIONS MODELS

ROGER E. A. FARMER DANIEL F. WAGGONER AND TAO ZHA

Abstract. We develop a new method for computing minimal state variable solutions (MSV) to Markov switching rational expectations models. We provide an algorithm to compute an MSV solution and we show how to test a given solution for uniqueness and boundedness. We construct an example, calibrated to US data, and we show that the MSV solution in our example is unique. This solution can potentially explain the observed reduction in the variance of inflation and the interest rate after 1980, in three different ways. The policy rule might have changed, the variance of the fundamental shocks might have fallen or the private sector equations might have been different across regimes. We compare these three explanations for the change in variance and we show that any one of them can potentially account for the facts. Our paper provides the necessary tools for a future empirical study of this issue.

I. Introduction

Following the introduction of vector autoregressions (VARs) to macroeconomics by Christopher Sims (1980) it was quickly realized that it is difficult to find macroeconomic applications for which model parameters remain stable over long periods of time. One approach to parameter instability, pursued by Richard Clarida, Jordi Galí and Mark Gertler (2000, CGG), and followed up by Thomas Lubik and Frank Schorfheide (2004, LS), is to break the sample into sub-periods and to estimate regime-dependent structural models in which one or more of the model's parameters are different across regimes. This is not entirely satisfactory since forward looking agents living in a world in which parameters are known to change occasionally would be expected to take possible parameter change into account when forming their expectations.

An alternative approach to parameter instability, suggested by the work of James Hamilton (1989) and pursued in a complete macro model by Christopher Sims and Tao Zha (2006), is to estimate a backward-looking vector autoregression (VAR) with regime dependent parameters. Their approach has its limitations since it does not allow for the presence of forward-looking components that are present in a dynamic stochastic general equilibrium (DSGE) model. To allow for change in the structural

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parameters of a model of this kind, we need a method for solving rational expectations models that incorporates regime change. This paper provides such a method.

Our approach is to expand the state space of a Markov switching rational expectations model (MSRE) and to write an equivalent model in this expanded space that has state-invariant parameters. We define a class of minimal state variable solutions (McCallum 1983, MSV) to the expanded model and we prove that any MSV solution is also a solution to the MSRE. We provide a set of necessary and sufficient conditions for the MSV solution to be unique and we show that MSV solutions can be characterized as a vector-autoregression with regime switching, of the kind studied by Hamilton (1989) and Sims and Zha (2006).

The class of solutions we identify is large, but it is not exhaustive. In a related paper (Farmer, Waggoner, and Zha 2006) we provide an example of a model with a unique MSV solution and we provide conditions under which this example also has a continuum of non MSV sunspot solutions.[1] In this paper, however, we study only the MSV solutions. The MSV solution is arguably the most interesting class to study since it is often stable under real time learning (Evans and Honkapohja 2001), (McCallum 2003).

In the second part of our paper, we provide an additional reason to be interested in MSV solutions to MSRE models. Clarida et al have argued that inflation, interest rates and output were all more volatile in the period before 1980 than after. Following CGG, it has become common to argue that the change in volatility observed at this time occurred *because* the Fed under Arthur Burns and William Miller followed a passive monetary policy that permitted the existence of sunspot equilibria. According to this interpretation of the data, the drop in volatility was a consequence of the removal of an additional variance term that, under Burns-Miller, was contributed by sunspots.

In Section XV, we present a calibrated New-Keynesian model of the kind estimated by Lubik and Schorfheide (2004) over two separate policy regimes. Unlike LS, we fix all of the parameters of the model except for the inflation coefficient on the Taylor rule and we show that this model possesses a unique MSV solution. Under our calibration, the move from a passive to an active regime leads to a reduction in the variance of output, inflation and the interest rate, of the same magnitude as observed in the data even when the standard deviation of all of the shocks is the same across regimes, as conjectured by Beyer-Farmer (2003). In contrast, Stock and Watson (2003) and Sims Zha (2006) studied a backward-looking Markov switching model and found no evidence in favor of changes in monetary policy. Instead, they attributed the change

[1] A sunspot solution is one where the state variables are functions of an extraneous random variable (Cass and Shell 1983). In Farmer, Waggoner, and Zha (2006), current state variables depend on expectations of future variables, but not on lagged state variables, and we show that there always exists a sunspot solution to the MSRE model when the parameters of the model in at least one regime would lead to indeterminacy if that regime were an absorbing state. This contradicts the results in Davig and Leeper (2005) who claim that determinacy of equilibrium depends on the probabilities contained in the Markov transition matrix.

in the time series behavior of the variables to time variation in shock variances. To examine the plausibility of this explanation we compare the policy change model with an alternative in which there are changes to the shock variances of the forward looking structural model.

II. How our Method Differs from Previous Approaches

Consider a structural linear rational expectations model with variables x_t and constant parameters, where x_t may include unobservable expectations conditional on date t information. The solution to a model of this kind can be described as a VAR in x_t. Now consider an extension that allows the structural parameters to be functions of a state ξ_t that is itself governed by a Markov chain. This is the model that we will study in this paper. Although our model is linear in x_t, it is non-linear in the complete set of state variables $\{x_t, \xi_t\}$ since its parameters, which are functions of ξ_t, enter the model multiplicatively with x_t.

Troy Davig and Eric Leeper (2005) and Davig, Leeper and Hess Chung (2003) have provided examples of models that are non-linear in both variables and parameters. The Davig-Leeper examples are more general than the linear-in-variables model that we consider here and, in certain special cases, they can be solved explicitly. However, the Davig-Leeper approach relies on special assumptions that cannot easily be generalized and even if their solution algorithm could be extended to a general class of non-linear variable-parameter rational expectations models, their method still does not provide a way of establishing that a proposed solution is unique or even that it is bounded and hence consistent with the transversality conditions of the structural model (Farmer, Waggoner, and Zha 2006).

Lars Svensson and Noah Williams (2005, SW) have developed an algorithm to compute a solution to a large class of linear-in-variables regime-switching models that is closely related to the class that we discuss in this paper. However, the SW solution method also lacks a diagnostic to inform the experimenter when the solution is unique and their algorithm may converge to a unique solution, to one of a set of indeterminate solutions, or to an unbounded stochastic difference equation that violates appropriate transversality conditions. In contrast, the method we describe in this paper computes a solution to a large class of variable-parameter linear rational expectations models and, within this class, it identifies whether the solution is unique, indeterminate or unstable. In this latter case, equilibrium (defined as a bounded stochastic process) fails to exist.[2]

Our approach builds on unpublished notes by Roger Farmer, Eric Leeper and Troy Davig (FLD) that describe a possible method for solving a linear rational expectations model by expanding the state space in a way that mimics existing techniques for

[2]Our method co-incides with the SW solution when the equilibrium is unique and has the added advantage of identifying the conditions for determinacy, indeterminacy and existence. In Appendix A, we show that the SW method will often converge to indeterminate equilibria or to unbounded solutions (non-existence).

finding a vector autoregressive representation of a Markov chain. The FLD approach, although promising, failed to maintain the important property that expectational errors must be mean zero. This omission is fatal because it gives incorrect results; remedying this error has proved to be a challenging task. This paper solves this problem and provides a way of converting a variable-parameter linear rational expectations model into an equivalent constant parameter model in a larger state space.

III. The Constant Parameter Model

We introduce the following class of linear models.

$$\overset{A}{\begin{bmatrix} \underset{(n-\ell)\times n}{a_1} \\ \underset{\ell\times n}{a_2} \end{bmatrix}} \underset{n\times 1}{x_t} = \overset{B}{\begin{bmatrix} \underset{(n-\ell)\times n}{b_1} \\ \underset{\ell\times n}{b_2} \end{bmatrix}} \underset{n\times 1}{x_{t-1}} + \overset{\Psi}{\begin{bmatrix} \underset{(n-\ell)\times k}{\psi} \\ \underset{\ell\times k}{0} \end{bmatrix}} \underset{k\times 1}{\varepsilon_t} + \overset{\Pi}{\begin{bmatrix} \underset{(n-\ell)\times \ell}{0} \\ \underset{\ell\times \ell}{\pi} \end{bmatrix}} \underset{\ell\times 1}{\eta_t} \tag{1}$$

where x_t is an $n\times 1$ set of endogenous variables, a_1, a_2, b_1, b_2, ψ, and π are conformable parameter matrices, ε_t is a $k \times 1$ vector of i.i.d. random variables and η_t is an $\ell \times 1$ vector of endogenous shocks, defined by the second ℓ rows of this system.

As an example, consider the model

$$y_t = aE_t\left[y_{t+1}|\Omega_t\right] + by_{t-1} + \sigma\varepsilon_t, \tag{2}$$

with matrices a_1, b_1, a_2, b_2, ψ and π defined as follows:

$$a_1 = \begin{bmatrix} 1 & -a \end{bmatrix},\ \ b_1 = \begin{bmatrix} b & 0 \end{bmatrix},$$
$$a_2 = \begin{bmatrix} 1 & 0 \end{bmatrix},\ \ b_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$
$$\psi = \sigma,\quad \pi = 1,$$

and

$$x_t = \begin{bmatrix} y_t \\ E_t\left[y_{t+1}\right] \end{bmatrix}.$$

The first $n-\ell$ equations of the model define the evolution of x_t as functions of lagged variables and fundamental shocks. The second ℓ equations define the endogenous errors. We partition the parameter matrices conformably and refer to the parameters of the first block of $n-\ell$ equations with the subscript 1 and to the second block of ℓ equations with the subscript 2.

A solution to Equation (1) is a bounded stochastic process $\{x_t\}$. Chris Sims (2001) provides a solution algorithm and sets of conditions on the matrices A, B, Ψ and Π under which there exists a unique solution, no solution or multiple solutions. Existence of a solution requires that the endogenous shock η_t can be adjusted every period to keep the system in a linear subspace for which solutions remain bounded and it depends on the properties of the matrices Ψ and Π and on the roots of the characteristic equation of the matrix pencil $|A - \lambda B| = 0$.[3] When a unique solution

[3]The roots of this characteristic equation are known as generalized eigenvalues of $\{A, B\}$. For a definition and discussion of the pencil of a quadratic form see Gantmacher (2000, Volume 1, page 310).

exists it has a representation in the form

$$x_t = \Gamma_1 x_{t-1} + \Gamma_2 \varepsilon_t. \tag{3}$$

Economic data often cannot be described by constant parameter models and James Hamilton (1989) amongst others, has suggested that reduced form models be described instead by VARs in which the parameters change occasionally. For example, the vector x_t might be described by the equation:

$$x_t = \Gamma_1 (\xi_t) x_{t-1} + \Gamma_2 (\xi_t) \varepsilon_t, \tag{4}$$

where ξ_t follows a Markov chain. Models of this kind have proved quite successful at describing the behavior of gdp and they have been applied by Sims and Zha (2002) to describe the behavior of inflation, interest rates and output. But how are reduced form Markov switching models related to structural rational expectations models? It is this question to which we now turn.

IV. The Markov Chain as an AR Process

The technique we will use to solve variable-parameter linear rational expectations models is based on an extension of the method, described in Hamilton (1994, page 679), by which a Markov chain can be described by an autoregressive process. In this section we explain this method and in Section V we show how it can be generalized to forward looking systems.

Let ξ_t be a two-state Markov chain with transition matrix P and define a pair of variables $z_{i,t}$, $i \in \{1, 2\}$ such that $z_{i,t} = \xi_t$ if $\xi_t = i$ and zero otherwise. Since the columns of P are conditional probability distributions the elements of P, denoted $[p_{ij}]$, satisfy the restriction

$$\sum_{i=1}^{2} p_{ij} = 1.$$

Let v_{it} be the difference between the realization of z_{it} and its conditional mean and write z_t as a vector-valued process:

$$z_t = P z_{t-1} + v_t, \tag{5}$$

where the vector v_t, defined as,

$$v_t = E (z_t | \Omega_{t-1}) - z_t, \tag{6}$$

has zero conditional mean. The conditional distribution of v_t is chosen to ensure that the state variable z_t satisfies the property,

$$z_{it} = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{if } \xi_t = 1 \\ \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \text{if } \xi_t = 2 \end{cases}, \quad \text{for all } t. \tag{7}$$

To keep z_t in the appropriate space, the conditional distribution of v_t must depend on the regime switching probabilities $[p_{ij}]$. This distribution is given by the expression;

$$\begin{bmatrix} v_{1t}(\xi_{t-1}=1) \\ v_{2t}(\xi_{t-1}=1) \end{bmatrix} = \begin{cases} \begin{bmatrix} 1-p_{11} \\ -p_{21} \end{bmatrix} & \text{if } \xi_t = 1, \\ \\ \begin{bmatrix} -p_{11} \\ 1-p_{21} \end{bmatrix} & \text{if } \xi_t = 2, \end{cases}$$

$$\begin{bmatrix} v_{1t}(\xi_{t-1}=2) \\ v_{2t}(\xi_{t-1}=2) \end{bmatrix} = \begin{cases} \begin{bmatrix} 1-p_{12} \\ -p_{22} \end{bmatrix} & \text{if } \xi_t = 1, \\ \\ \begin{bmatrix} -p_{12} \\ 1-p_{22} \end{bmatrix} & \text{if } \xi_t = 2. \end{cases}$$

The shock vector v_t has zero conditional mean since

$$\begin{aligned} E_{t-1}[v_t|\xi_{t-1}=i] &= p_{1i}\begin{bmatrix} 1-p_{1i} \\ -p_{2i} \end{bmatrix} + p_{2i}\begin{bmatrix} -p_{1i} \\ 1-p_{2i} \end{bmatrix}, \\ &= p_{1i}\begin{bmatrix} 1-p_{1i} \\ -(1-p_{1i}) \end{bmatrix} + (1-p_{1i})\begin{bmatrix} -p_{1i} \\ p_{1i} \end{bmatrix}, \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

The following sections show how to extend the idea that a Markov chain has an autoregressive representation to the general linear rational expectations model. In the extension we will exploit the fact that the autoregressive representation has constant state-independent parameters to find a solution method for regime switching models with rational expectations.

V. The Markov-Switching Model

This section introduces a class of Markov-switching rational expectations models and in Sections VIII–XI we show how to compute a minimal state variable solution to models in this class by expanding the state space. In Section XII we present formal definitions and we prove that a solution to the expanded model is also a solution to the original Markov switching model. We discuss the properties of equilibrium and we show that the conditions governing existence and uniqueness are more complex than in the case of single regime models. We propose a class of minimum state variable solutions (McCallum 1983, MSV) and we demonstrate that when a unique equilibrium exists, it is in this class. Our algorithm can also identify an MSV solution to models with indeterminate equilibria (when one exists) and it produces a diagnostic that alerts the researcher to the presence of multiplicity or to the non-existence of a bounded solution.

Consider the following generalization of Equation (1),

$$\overset{A(\xi_t)}{\begin{bmatrix} \underset{(n-\ell)\times n}{a_1(\xi_t)} \\ \underset{\ell\times n}{a_2(\xi_t)} \end{bmatrix}} \underset{n\times 1}{x_t} = \overset{B(\xi_t)}{\begin{bmatrix} \underset{(n-\ell)\times n}{b_1(\xi_t)} \\ \underset{\ell\times n}{b_2(\xi_t)} \end{bmatrix}} \underset{n\times 1}{x_{t-1}} + \overset{\Psi(\xi_t)}{\begin{bmatrix} \underset{(n-\ell)\times k}{\psi_1(\xi_t)} \\ \underset{\ell\times k}{\psi_2(\xi_t)} \end{bmatrix}} \underset{k\times 1}{\varepsilon_t} + \overset{\Pi(\xi_t)}{\begin{bmatrix} \underset{(n-\ell)\times \ell}{\pi_2(\xi_t)} \\ \underset{\ell\times \ell}{\pi_2(\xi_t)} \end{bmatrix}} \underset{\ell\times 1}{\eta_t}, \tag{8}$$

$$x_0 = \bar{x}_0, \tag{9}$$

where ξ_t follows an $h-$state Markov chain, $h \in H \equiv \{1, ...h\}$, with stationary transition matrix P and η_t is a mean zero endogenous process. The matrix $\Pi(\xi_t)$ is assumed to have full rank, and thus without loss of generality we take $\pi_1(\xi_t) = 0$, $\pi_2(\xi_t) = I_\ell$, $\psi_1(\xi_t) = \psi(\xi_t)$, and $\psi_1(\xi_t) = 0$, where I_ℓ is the $\ell \times \ell$ identity matrix. In most applications it will be the case that x_t is partitioned as

$$x_t = \begin{bmatrix} y_t \\ z_t \\ E_t y_{t+1} \end{bmatrix}, \tag{10}$$

where y_t is of dimension ℓ and the second block of Equation (8) is of the form $y_t = E_{t-1}y_t + \eta_t$. Though we do not impose this restriction in this paper, it is useful to keep this example in mind when interpreting our results.

VI. Minimum State Variable Solutions

In the constant parameter case, solutions of Equation (1) are obtained by restricting the solution x_t to lie in some linear subspace of $\mathbb{R}^n$. The potential subspaces are determined by the generalized eigenvalue-eigenvector decomposition of A and B.[4] For instance, if the solution is restricted to the subspace is spanned by the eigenvectors associated with the eigenvalues that are less than one in absolute value, then one obtains the stable manifold. This is the largest subspace that supports bounded solutions, assuming, of course, that bounded solutions exist. If one considers the partition of x_t given by Equation (10), then restricting the solution to lie in some linear subspace forces linear relations among $E_t y_{t+1}$, y_t and z_t. A MSV solution is one in which enough restrictions have been imposed so that $E_t y_{t+1}$ is completely determined and so there can be no sunspot shocks, but not so many restrictions that y_t and z_t are constrained. We do not want to constrain y_t and z_t because we would like the solution to exist for any initial conditions of y_t and z_t. In the constant parameter case this means that the potential subspaces are generated by any set of $n-\ell$ eigenvectors whose eigenvalues are all less than one in absolute value.

[4]If x_t is a solution of Equation (1) and the span of the support of x_t over all t is the linear subspace $\hat{V}$, then it must be the case that $\hat{V}$ is generated by some subset of the generalized eigenvectors of A and B. However, it is not necessarily the case any subspace generated by a subset of the generalized eigenvectors of A and B will support a solution of Equation (1). It is in this sense that the eigenvectors only generate potential subspaces that support solutions. Existence conditions must always be confirmed.

These ideas can be generalized to the time varying parameter case. Define $1_{\{s_t=i\}}$ to be indicator function which is one if $s_t = i$ and zero otherwise and define the span of x_t, conditional on $s_t = i$, to be the linear span of the support $x_t 1_{\{s_t=i\}}$ over all t. Again, considering the partition of x_t given by Equation (10), we want to impose enough restrictions so that $E_t y_{t+1}$ is completely determined and so there can be no sunspot shocks, but not so many that y_t and z_t are constrained. This implies that the dimension of each conditional span must be $n-s$. Unlike the constant parameter case, there is not an easy to apply eigenvalue condition that allows us to identify all potential candidates for the conditional spans and so we must resort to iterative techniques to find MSV solutions. However, we do have the following characterization of any MSV solution.

Theorem 1. If $\{x_t, \eta_t\}_{t=1}^{\infty}$ is a MSV solution of Equation (8), then

$$x_t = V_{\xi_t} F^1_{\xi_t} x_{t-1} + V_{\xi_t} G^1_{\xi_t} \varepsilon_t, \tag{11}$$

$$\eta_t = -\left(F^2_{\xi_t} x_{t-1} + G^2_{\xi_t} \varepsilon_t\right), \tag{12}$$

where V_j is $n \times (n-\ell)$, F^1_j is $(n-\ell) \times n$, F^2_j is $\ell \times n$, G^1_j is $(n-\ell) \times k$, and G^2_j is $\ell \times k$. Furthermore, $\begin{bmatrix} A(j)V_j & \Pi \end{bmatrix}$ is invertible and

$$\begin{bmatrix} A(j)V_j & \Pi \end{bmatrix} \begin{bmatrix} F^1_j \\ F^2_j \end{bmatrix} = B(j), \tag{13}$$

$$\begin{bmatrix} A(j)V_j & \Pi \end{bmatrix} \begin{bmatrix} G^1_j \\ G^2_j \end{bmatrix} = \Psi(j), \tag{14}$$

$$\left(\sum_{j=1}^{h} p_{i,j} F^2_j\right) V_i = 0_{\ell, n-\ell}. \tag{15}$$

This theorem is proved in the appendix, but it is worth noting that Conditions (13) and (14) ensure that $\{x_t, \eta_t\}_{t=1}^{\infty}$ given by Equations (11) and (12) satisfy Equation (8), while Condition (15) ensures that $E_{t-1}\eta_t = 0$. This theorem will form the basis for the iterative techniques considered in this paper.

VII. Newton's Method

In this section, we show how Newton's Method, or any other root finding algorithm, can be used to find MSV solutions of Equation (8). To make the analysis more transparent, we assume that $A(j)$ is invertible. In the appendix, we show how to relax that condition. To find a MSV solution, we must find matrices V_j such that $\begin{bmatrix} A(j)V_j & \Pi \end{bmatrix}$ is invertible and Equation (15) holds where F^2_j is defined via Equation (13). Since $\Pi' = [0_{\ell,n-\ell} \;\; I_\ell]$, and V_j is only defined up to right multiplication by an invertible matrix, we can assume that

$$A(j)V_j = \begin{bmatrix} I_{n-\ell} \\ -X_j \end{bmatrix} \tag{16}$$

for some $\ell \times (n-\ell)$ matrix X_j. Since

$$\begin{aligned} F_j^2 &= \begin{bmatrix} 0_{\ell,n-\ell} & I_\ell \end{bmatrix} \begin{bmatrix} A(j)V_j & \Pi \end{bmatrix} B(j) \\ &= \begin{bmatrix} X_j & I_\ell \end{bmatrix} B(j), \end{aligned}$$

Equation (15) becomes

$$\sum_{j=1}^{h} p_{i,j} \begin{bmatrix} X_j & I_\ell \end{bmatrix} B(j) A(i)^{-1} \begin{bmatrix} I_{n-\ell} \\ -X_i \end{bmatrix} = 0_{\ell,n-\ell}. \tag{17}$$

So, if we define f_i to be the function from $\mathbb{R}^{h\ell(n-\ell)}$ to $\mathbb{R}^{\ell(n-\ell)}$ given by

$$f_i(X_1, \cdots, X_h) = \sum_{j=1}^{h} p_{i,j} \begin{bmatrix} X_j & I_\ell \end{bmatrix} B(j) A(j)^{-1} \begin{bmatrix} I_{n-\ell} \\ -X_j \end{bmatrix} \tag{18}$$

and f to be the function from $\mathbb{R}^{h\ell(n-\ell)}$ to $\mathbb{R}^{h\ell(n-\ell)}$ given by

$$f(X_1, \cdots, X_h) = (f_1(X_1, \cdots, X_h), \cdots, f_h(X_1, \cdots, X_h)), \tag{19}$$

finding MSV solutions is equivalent to finding the roots of $f(X_1, \cdots, X_h)$. This can be accomplished using any root finding technique. In the appendix, we give the details for applying Newton's method to find the roots of this equation. In the next several sections, we outline a different iterative technique for finding MSV solutions.

VIII. Expanding the State Space

In Sections VIII, IX, and X we assume the existence of an MSV solution to Equation (8) and we show how to write an equivalent model that is linear in variables and has constant parameters. Section XI shows the relationship between the equivalent model and the original MSRE model. In these sections we make the additional mild assumption that $A_2(j)$ and $B_2(j)$ in Equation (8) do not depend on j. In almost all economic applications, this will be the case.

First, we write Equation (8) as follows,

$$A(\xi_t) x_t = B(\xi_t) x_{t-1} + \Psi(\xi_t) \varepsilon_t + \Pi(\xi_t) \eta_t, \tag{20}$$

where

$$A(i) = \begin{bmatrix} a_1(i) \\ a_2 \end{bmatrix}, B(i) = \begin{bmatrix} b_1(i) \\ b_2 \end{bmatrix}, \tag{21}$$

$$\Psi(i) = \begin{bmatrix} \psi(i) \\ 0 \end{bmatrix} \text{ and } \Pi(i) = \begin{bmatrix} 0 \\ \pi \end{bmatrix}. \tag{22}$$

We now show how to define an expanded state state vector X_t and constant parameter matrices A, B, Ψ and Π such that system (20) can be written as follows,

$$AX_t = BX_{t-1} + \Psi u_t + \Pi \eta_t. \tag{23}$$

We are interested in the existence and uniqueness of an MSV solution to (20) and for this purpose Equation (23) is a useful way of representing the model because it has constant parameter matrices, A, B, Ψ and Π and because the shock vectors u_t and

η_t have zero means. These properties allow us to use known techniques to compute a solution.

IX. Definitions of the Parameter Matrices

We begin by assuming the existence of a family of matrices $\{\phi_i\}_{i=1}^{h}$ where each ϕ_i has dimension $\ell \times n$ and has full row rank. Later, we will show how to express $\{\phi_1, \phi_2, ...\phi_h\}$ as the fixed point of a system of nonlinear equations. Define the matrix Φ as follows,

$$\underset{(\ell-1)h\times nh}{\Phi} = \begin{bmatrix} \mathbf{e}_2' \otimes \phi_2 \\ \vdots \\ \mathbf{e}_h' \otimes \phi_h \end{bmatrix}, \tag{24}$$

and let the matrices A, B, and Π be given by the expressions,

$$\underset{nh\times nh}{A} = \begin{bmatrix} diag\left(a_1\left(1\right), \cdots, a_1\left(h\right)\right) \\ a_2 \quad \cdots \quad a_2 \\ \Phi \end{bmatrix}, \tag{25}$$

$$\underset{nh\times nh}{B} = \begin{bmatrix} diag\left(b_1\left(1\right), \cdots, b_1\left(h\right)\right)\left(P \otimes \mathbf{I}_n\right) \\ b_2 \quad \cdots \quad b_2 \\ 0 \end{bmatrix}, \tag{26}$$

$$\underset{nh\times \ell}{\Pi} = \begin{bmatrix} 0 \\ \pi \\ 0 \end{bmatrix}. \tag{27}$$

X. Definitions of the Shocks

We now introduce definitions of the shocks u_t and η_t. Let $\mathbf{I}_n$ denote the $n \times n$ identity matrix, let $\mathbf{e}_i$ denote the i^{th} column of $\mathbf{I}_h$, and let $\mathbf{1}_h$ be the h-dimensional column vector of ones. Let $X_t \in R^{nh}$ be an arbitrary vector of dimension $nh \times 1$.

Next define h matrices S_i for $i \in \{1, ...h\}$ as follows

$$\underset{(n-\ell)h\times nh}{S_i} = \left(diag\left[b_1\left(1\right), \cdots, b_1\left(h\right)\right]\right) \times \left[\left(\mathbf{e}_i \mathbf{1}_h' - P\right) \otimes \mathbf{I}_n\right], \tag{28}$$

Equation (28) generates a matrix of dimension $(n-\ell)\,h \times nh$ where $b_1(i)$ has dimension $(n-\ell) \times n$ and P is $h \times h$. S_i consists of h^2 blocks each of which is $(n-\ell) \times n$. For $j \neq i$ the $j'th$ row-block of S_i contain terms of the form $-p_{jk} b_1(j)$. The $i'th$ row-block contains terms of the form $(1 - p_{ik})\, b_1(i)$.

Using this notation, we define the vector of error terms

$$u_t = \begin{bmatrix} S_{\xi_t}\left(\mathbf{e}_{\xi_{t-1}} \otimes \left(\mathbf{1}_h' \otimes \mathbf{I}_n\right) X_{t-1}\right) \\ \mathbf{e}_{\xi_t} \otimes \varepsilon_t \end{bmatrix}, \tag{29}$$

and the matrix Ψ as follows

$$\underset{nh\times(k+n-\ell)h}{\Psi} = \begin{bmatrix} \mathbf{I}_{(n-\ell)h} & diag\left(\psi\left(1\right), \cdots, \psi\left(h\right)\right) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{30}$$

The error term u_t contains two kinds of shocks that we refer to as *switching* shocks and *normal* shocks. The switching shocks, represented by the terms

$$S_{\xi_t}\left(\mathbf{e}_{\xi_{t-1}} \otimes \left(\mathbf{1}_h' \otimes \mathbf{I}_n\right) X_{t-1}\right), \tag{31}$$

turn on or off the appropriate blocks of the model to represent the Markov dynamics. The normal shocks, represented by the terms

$$\mathbf{e}_{\xi_t} \otimes \varepsilon_t, \tag{32}$$

carry the fundamental errors that hit the structural equations, distributed to the appropriate block of the expanded system.

It is important for our solution technique that the errors have mean zero. Since ξ_t is uncorrelated with ε_t, and since ε_t has mean zero, the normal shocks satisfy this property. Note also that

$$\begin{aligned} &E_{t-1}\left[S_{\xi_t}\left(\mathbf{e}_{\xi_{t-1}} \otimes \left(\mathbf{1}_h' \otimes \mathbf{I}_n\right) X_{t-1}\right)\right] \\ &\qquad = diag\left(b_1(1), \cdots, b_1(h)\right)\left(\left(E_{t-1}\left[\mathbf{e}_{\xi_t}\right] - P\mathbf{e}_{\xi_{t-1}}\right) \otimes \left(\mathbf{1}_h' \otimes \mathbf{I}_n\right) X_{t-1}\right) = 0, \end{aligned} \tag{33}$$

where the equality follows from the fact that

$$E_{t-1}\left[\mathbf{e}_{\xi_t}\right] = \begin{bmatrix} p_{1i} \\ \vdots \\ p_{hi} \end{bmatrix} = P\mathbf{e}_{\xi_{t-1}}.$$

Hence, $E_{t-1}\left[u_t\right] = 0$.

XI. The Relationship between the Expanded and the MSRE Form

Suppose now that X_{t-1} has the additional structure, $X_{t-1} = \mathbf{e}_{\xi_{t-1}} \otimes x_{t-1}$ and partition X_t' as $X_t' = \left[x_{1,t}', \cdots, x_{h,t}'\right]$. Now partition Equation (23) into three blocks corresponding to the first $(n-\ell)h$ rows the next ℓh rows and the last $(\ell-1)h$ rows. This partition corresponds to the three row blocks in Equations (25), (26) and (27). Using the definitions of A, B, Ψ, Π, u_t and η_t, the first block of Equation (23) reduces to the expression

$$a_1(i)\, x_{i,t} = \begin{cases} 0 & \text{for } i \neq \xi_t \\ b_1(i)\, x_{t-1} + \psi(i)\, \varepsilon_t & \text{for } i = \xi_t \end{cases}, \tag{34}$$

the second block reduces to

$$a_2 \sum_{i=1}^{h} x_{i,t} = b_2 x_{t-1} + \pi \eta_t, \tag{35}$$

and the third block gives us the equation,

$$\phi_i x_{i,t} = 0 \text{ for } 1 < i \leq h. \tag{36}$$

It follows that, when X_t has this additional structure, Equations (34) and (35) reproduce Equation (20) hence, if $\{X_t, \eta_t\}_{t=1}^{\infty}$ is consistent with Equation (23) then $\{x_t, \eta_t\}_{t=1}^{\infty}$ must be consistent with the original system. The remaining question is concerned with whether the sequence $\{x_t\}$ is bounded and here, we will need to be more precise about the role of the matrices ϕ_i in the expanded system.

XII. Definitions and Theorems

We begin by defining formally what it means for a stochastic process to be a solution to a Markov rational expectations model. We define a stochastic process $\{x_t, \eta_t\}$ to be a solution to the model if it satisfies Equation (20) and if, in addition, the solution remains bounded in expectation.

Definition 1 (Solution). A solution to Equation (20) is a stochastic process $\{x_t, \eta_t\}_{t=1}^{\infty}$ such that:
(1) The endogenous stochastic process $\{\eta_t\}$ satisfies the property, $E_{t-1}[\eta_t] = 0$.
(2) x_t is bounded in expectation in the sense that $\|E_t[x_{t+s}]\| < M_t$ for all $s > 0$.
(3) $\{x_t, \eta_t\}_{t=1}^{\infty}$ jointly satisfy Equation (20).

The complete class of solutions to an MSRE model is very rich and our computational experiments with these models suggests that sunspot solutions are pervasive. In this paper we will concentrate on a class of minimal state variable solutions which we define as follows.

Definition 2 (MSV Solution). Let $\{x_t, \eta_t\}_{t=1}^{\infty}$ be a solution to Equation (20). $\{x_t, \eta_t\}_{t=1}^{\infty}$ is a minimum state variable solution if and only if:
(1) There exist minimal state matrices χ_i for $1 \le i \le h$ such that $\chi_{\xi_t} x_t = 0$ for all $t \ge 0$.
(2) If $\{\tilde{x}_t, \tilde{\eta}_t\}_{t=1}^{\infty}$ is any solution to Equation (20) such that $\tilde{\chi}_{\xi_t} x_t = 0$ for all $t \ge 0$ and the row space of $\tilde{\chi}_i$ contains the row space of χ_i, then the row space of $\tilde{\chi}_i$ equals the row space of χ_i.

Our definition is consistent with standard usage, for example, Evans and Honkapohja define a minimal state variable solution of a linear difference equation to be one which "depends linearly on a set of variables...such that there does not exist a solution which depends linearly on a smaller set of variables" (Evans and Honkapohja 2001, page 176). Condition (2) states that the minimal states matrices are as large as possible and hence the variables are restricted to subspace whose dimension is as small as possible.

Our first theorem concerns the relationship between MSV solutions to the expanded model and MSV solutions to the original model. Note that our definition of a minimal state variable solution applies when $h = 1$ and hence the MSV solution to Equation (23) is well defined.

Theorem 2. Suppose $\{x_t, \eta_t\}_{t=1}^{\infty}$ is a minimum state variable solution of Equation (20) and let χ_i, for $1 \le i \le h$, be the associated minimal state matrices. Suppose further that χ_i is of dimension $q_i \times n$, and $\sum_{i=1}^{h} q_i \ge \ell(h-1)$. For $1 \le i \le h$, let ϕ_i be any

$q_i \times n$ submatrix of χ_i such that $\sum_{i=1}^{h} q_i = \ell(h-1)$. If

$$\Phi = \begin{bmatrix} \mathbf{e}_1 \otimes \phi_1 \\ \vdots \\ \mathbf{e}_h \otimes \phi_h \end{bmatrix},$$

then $X_t = \mathbf{e}_{\xi_t} \otimes x_t$ is a solution of Equation (23).

Proof. See Appendix. □

If we know the matrices χ_i associated with an MSV solution to Equation (20) then Theorem 2 tells us that there will typically be many ways to pick the matrix Φ that is needed to define A in Equation (23). For example, if $q_i = \ell$ for all $i \in H$ then one possible choice for Φ is to set ϕ_1 to the empty matrix and choose $\phi_i = \chi_i$ for $i = 2, ...h$. This would be consistent with the definition of Φ given in Equation (24). More generally, this definition implies that there are other ways of defining Φ that do not conform to the structure of Equation (24).

So far we have said nothing about how to find the matrix Φ. The following definition of the unstable component of $\{A, B\}$ is designed to lead up to a theorem that enables us to compute Φ.

Definition 3. Let $QSZ = A$ and $QTZ = B$ be the $QZ-$decomposition of $\{A, B\}$, where the upper triangular matrices $S = (s_{i,j})$ and $T = (t_{i,j})$ have been arranged so that $t_{i,i}/s_{i,i}$ is in are in increasing order.[5] Let $q \in \{1, 2..., h\}$ be the integer such that $t_{i,i}/s_{i,i} < 1$ if $i \leq q$ and $t_{i,i}/s_{i,i} > 1$ if $i > q$. Let Z_u, partitioned as $Z_u = [z_1, \cdots, z_h]$, be the last $nk - q$ rows of Z. We refer to Z_u as the unstable component of $\{A, B\}$.

Theorem 3. If $\begin{bmatrix} a_1'(i) & z_i' \end{bmatrix}$ has rank n for $1 \leq i \leq h$ and the row space of z_i is equal to the row space of ϕ_i for $2 \leq i \leq h$, then any solution $\{X_t, \eta_t\}_{t=1}^{\infty}$ of Equation (23) with $X_0 = e_{\xi_0} \otimes x_0$ will be of the form $X_t = e_{\xi_t} \otimes x_t$ for all t, where $\{x_t, \eta_t\}_{t=1}^{\infty}$ is a solution of Equation (20).

Proof. See Appendix. □

Theorem 2 asserts an equivalence between an MSV solution to the original model and an MSV solution to Equation (23), but since $\{\chi_i\}_{i=1}^{h}$ and Φ are both unknown this theorem is not of much help in computing a solution. In contrast, Theorem 3 suggests a way to compute Φ. For given Φ, we can define A and given $\{A, B\}$ we can compute $Z_u = [z_1, \cdots, z_h]$. In general the row space of z_i, computed in this way, will not be equal to the row space of ϕ_i. However, in practice it is possible to define $\{\phi_i\}$ as a fixed point of a particular nonlinear map and to compute this fixed point iteratively. Theorem 3 gives conditions which ensure that a solution of the expanded system is a solution of the original system. The next two theorems give some properties of the solutions obtained from the expanded system.

[5]The reader is referred to Gene Golub and Charles Van Loan (1996, page 375) for a discussion of the QZ decomposition and to Sims (2001) for the application of the QZ method to solve linear rational expectations models.

Theorem 4. Suppose $\left[\begin{array}{cc} a_1'(i) & z_i' \end{array}\right]$ has rank n for $1 \leq i \leq h$ and the row space of z_i is equal to the row space of ϕ_i for $2 \leq i \leq h$. If there exists an $\ell \times k$ matrix Λ such that $Z_2(\Psi + \Pi\Lambda) = 0$, then there exists a solution to Equation (20) and this is the only solution of Equation (20) that satisfies the linear restriction $\phi_{\xi_t} x_t = 0$ for all $t > 0$.

Proof. See Appendix. □

Recall that the definition of a solution includes the restriction of boundedness. It is easy to find arbitrary mean zero sequences $\{\eta_t\}_{t=1}^{\infty}$ that generate a stochastic process $\{x_t\}_1^{\infty}$, consistent with Equation (20). But these sequences will not in general be bounded in expectation. Theorem 4, based on Lemma 1 from Lubik and Schorfheide (2003)[page 277], provides a condition under which one can find a sequence $\{\eta_t\}_{t=1}^{\infty}$ which ensures that $\|E_t[x_{t+s}]\|$ is bounded.

When a bounded solution exists, the matrix Λ defines the relationship between the fundamental shocks ε_t and the endogenous errors η_t. These shocks are related to each other by the equation

$$\eta_t = \Lambda\varepsilon_t. \tag{37}$$

The final theorem in this section provides an answer to the question: How are reduced form Markov switching models related to MSRE models?

Theorem 5. Suppose that $\{X_t, \eta_t\}_{t=1}^{\infty}$ is a solution of Equation (23) and $X_t = \mathbf{e}_{\xi_t} \otimes x_t$ for all $t \geq 0$. Then

$$x_t = g_1(\xi_t) x_{t-1} + g_2(\xi_t)\varepsilon_t.$$

Proof. See Appendix. □

XIII. How to Compute an MSV Solution and Check if it is Unique

Theorems 3, 4 and 5 work together in the following way. Let the superscript n on a variable denotes the $n'th$ step of an iterative procedure. Beginning with a set of matrices $\{\phi_i^0\}_{i=2}^{h}$, define Φ^0 using Equation (24) and generate the associated matrix A^0. Next, calculate Z_u^0 by computing the QZ decomposition of $\{A^0, B\}$ and set $\phi_i^1 = z_i^1$. This leads to a new matrix A^1 and a new set of values for ϕ_i^1. repeat this procedure and, if it converges, Theorem 3 implies that Equation (23) generates sequences $\{x_t, \eta_t\}_{t=1}^{\infty}$ that are consistent with Equation (20). To check that $\{x_t\}_{t=1}^{\infty}$ is bounded in expectation apply the procedure described by Sims (2001) or Harald Uhlig (1999) to check for existence and or uniqueness of a solution to Equation (23). If a unique solution exists, Sim's procedure generates the matrix Λ from Theorem 4 and Theorem 5 implies that the solution can be written as a VAR with time dependent coefficients.

We end with a word of warning. Although our method identifies the unique MSV solution, if one exists, MSRE models often have many other solutions. In Farmer et al (2006) we construct an example of a purely forward looking model and we parameterize it in two ways. The first leads to the existence of a unique solution and the second to multiple indeterminate solutions. We then consider an MSRE model with an arbitrary transition matrix that switches between the two forward looking models. In this environment we prove that the MSRE model always has a continuum of sunspot equilibria whatever values we choose for the transition probabilities. Since our example contains a unique MSV solution we conclude that uniqueness of the MSV solution does not imply uniqueness in a larger class of solutions.

XIV. An Illustrative Example

To illustrate how to use our method, consider the following simple dynamic equilibrium model

$$y_t = a(\xi_t) E_t y_{t+1} + b(\xi_t) y_{t-1} + \epsilon_t, \tag{38}$$

where $\xi_t \in \{1, 2\}$ and ϵ_t is normally distributed with mean zero and variance $\sigma^2(\xi_t)$. Here $m = 2$, $n = 2$ and $l = 1$. If the MSV solution to (38) exists, it will have the form

$$y_t = g_1(\xi_t) y_{t-1} + g_2(\xi_t)\epsilon_t. \tag{39}$$

To obtain the solution (39), we propose to write the original model (38) in the following expanded form,

$$AX_t = BX_{t-1} + \Psi u_t + \Pi \eta_t, \tag{40}$$

where $\eta_t = y_t - E_{t-1} y_t$.

To construct A, B, Ψ, and u_t, we define the matrices A_1 and B_1 as

$$A_1 = \begin{bmatrix} 1 & -a(1) & 0 & 0 \\ 0 & 0 & 1 & -a(2) \end{bmatrix},$$

$$B_1 = \begin{bmatrix} p_{11} b(1) & 0 & p_{12} b(1) & 0 \\ p_{21} b(2) & 0 & p_{22} b(2) & 0 \end{bmatrix},$$

and the state, X_t, by

$$X_t = \begin{bmatrix} x_{11,t} \\ x_{12,t} \\ x_{21,t} \\ x_{22,t} \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{\iota}_{(\xi_t=1)} y_t \\ \boldsymbol{\iota}_{(\xi_t=1)} E_t y_{t+1} \\ \boldsymbol{\iota}_{(\xi_t=2)} y_t \\ \boldsymbol{\iota}_{(\xi_t=2)} E_t y_{t+1} \end{bmatrix}.$$

Let

$$x_{i,t} = \begin{bmatrix} x_{i1,t} \\ x_{i2,t} \end{bmatrix},$$

for $i = 1, 2$. If $\xi_{t-1} = 1$, then $x_{2,t-1} = 0$; if $\xi_{t-1} = 2$, $x_{1,t-1} = 0$. It follows that either the first or the third column of B_1 will be zeroed out by X_{t-1}. If $\xi_{t-1} = j$, the nonzero element is $p_{1j} b(1)$ in the first row and $p_{2j} b(2)$ in the second row. The coefficient matrices A and B are constructed as

$$A_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & \phi_1 & \phi_2 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

We now show how to construct the vector of fundamental shocks u_t and the matrix Ψ in such a way that one of the first two rows is always zeroed out (by the shock) and the remaining row is set equal to $b\,(i)$ when $\xi_t = i$. Using the notation introduced in Section X, we have

$$\underset{2\times 4}{S_{\xi_t}} = \operatorname{diag}([b(1)\ 0],:[b(2)\ 0])\ [(\mathbf{e}_{\xi_t} \otimes \mathbf{1}_2' - P) \otimes \mathbf{I}_2]\,,$$
$$= \begin{bmatrix} (\boldsymbol{\iota}_{(\xi_t=1)}) - p_{11})\, b(1) & 0 & (\boldsymbol{\iota}_{(\xi_t=1)}) - p_{12})\, b(1) & 0 \\ (\boldsymbol{\iota}_{(\xi_t=2)}) - p_{21})\, b(2) & 0 & (\boldsymbol{\iota}_{(\xi_t=2)}) - p_{22})\, b(2) & 0 \end{bmatrix},$$
$$\underset{6\times 1}{u_t} = \begin{bmatrix} \begin{bmatrix} \underset{2\times 4}{S_{\xi_t}} \\ \underset{2\times 4}{\mathbf{0}} \end{bmatrix} \underset{4\times 1}{X_{t-1}} \\ \mathbf{e}_{\xi_t} \otimes \epsilon_t \end{bmatrix},\ \underset{4\times 6}{\Psi} = \begin{bmatrix} \underset{2\times 2}{\mathbf{I}} & \underset{2\times 2}{\mathbf{0}} & \underset{2\times 2}{\mathbf{I}} \\ \underset{2\times 2}{\mathbf{0}} & \underset{2\times 2}{\mathbf{0}} & \underset{2\times 2}{\mathbf{0}} \end{bmatrix},\ \Pi = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

The last equation in the expanded system (40) takes care of the zero restrictions in regime 1, but we also need to make sure that when regime 2 occurs, $x_{11,t}$ and $x_{12,t}$ are set to zero. To accomplish this, our method constrains the the solution X_t to lie in the stable linear subspace. This constraint is imposed by defining a vector $Z = [z_1\ z_2\ z_3\ z_4]'$ such that

$$Z' X_t = 0. \tag{41}$$

Let $\phi_1 = z_3$ and $\phi_2 = z_4$. When regime 2 occurs at date t, Equation (41) restricts the vector $\{y_t, E_t y_{t+1}\}$ and, in conjunction with the last equation of (40), it ensures that a linear combination of $x_{11,t}$ and $x_{12,t}$ is equated to zero:

$$\begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} x_{11,t} \\ x_{12,t} \end{bmatrix} = 0.$$

A second linear combination of these variables is set to to zero by the first two equations in (40). Hence, our method switches off the variables associated with the regime that does not occur and it switches on the appropriate equations that govern the evolution of y_t and $E_t y_{t+1}$ in the regime that does occur.

Theorem 3 ensures that the solution to (40) is also the solution to the original switching model (38). Consider an example in which $a(1) = 0.3$, $a(2) = 0.7$, $b(1) = 0.2$, and $b(2) = 0.5$. If only one regime were to exist, the first choice of parameter values would cause this regime to be determinate and the second would cause it to be indeterminate. If the transition matrix is

$$P = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix},$$

our method indicates that there is no MSV solution to (40). If the transition matrix is

$$P = \begin{bmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{bmatrix},$$

there exists a unique MSV solution for which $g_1(1) = 0.2187$, $g_1(2) = 0.8757$, $g_2(1) = 1.0933$, and $g_2(2) = 1.7513$. In summary, this example shows that the existence and uniqueness of an MSV solution to (40) depends, in general, on the values of the model parameters in each regime and on the transition matrix P.

XV. An Application to A Monetary Policy Model

In this section we present simulation results based on a calibrated version of the New-Keynesian model and we use it to study changes in output, inflation, and the nominal interest rate. Clarida, Galí, and Gertler (2000) and Lubik and Schorfheide (2004) argue that the large fluctuations in output, inflation, and interest rates are caused by indeterminacy led by passive monetary policy. Sims and Zha (2006), on the other hand, find no evidence in favor of indeterminacy when they allow monetary policy to switch regimes stochastically. Furthermore, they find that once the model permits time variation in disturbance variances, there is no evidence in favor of policy changes at all (see also Cogley and Sargent (2005), Giorgio Primiceri (2005) and Uhlig (1997)).

Once it is known that policy changes might occur, a *rational* agent should treat these changes probabilistically and the probability of a future policy change should enter into his current decisions. Previous work in this area has neglected these effects and all of the studies cited above study regime switches in a purely reduced form model. We show in this section how to use the MSV solution to a MSRE model to study the effects of regime change that is rationally anticipated to occur. We use simulation results to show that the persistence and volatility in inflation and the interest rate can be the result of (1) policy changes, (2) changes in shock variances, or (3) changes in private sector parameters. Hence, our method provides a tool for empirical work, in which a more formal analysis of the data can be used to discriminate between these competing explanations.

Our regime-switching policy model, based on Lubik and Schorfheide (2004), has the following three structural equations:

$$x_t = E_t x_{t+1} - \tau(\xi_t)(R_t - E_t \pi_{t+1}) + z_{D,t}, \tag{42}$$

$$\pi_t = \beta(\xi_t) E_t \pi_{t+1} + \kappa(\xi_t) x_t + z_{S,t}, \tag{43}$$

$$R_t = \rho_R(\xi_t) R_{t-1} + (1 - \rho_R(\xi_t)) \left[\gamma_1(\xi_t)\pi_t + \gamma_2(\xi_t) x_t\right] + \epsilon_{R,t}, \tag{44}$$

where x_t is the output gap at time t, π_t is the inflation rate and R_t is the nominal interest rate. Both π_t and R_t are measured in terms of deviations from the regime-dependent state. The coefficient τ measures the intertemporal elasticity of substitution; β is the household's discount factor and the parameter κ reflects the rigidity or stickiness of prices.

The shocks to the consumer and firm's sectors, $z_{D,t}$ and $z_{S,t}$, are assumed to evolve according to an AR(1) process:

$$\begin{bmatrix} z_{D,t} \\ z_{S,t} \end{bmatrix} = \begin{bmatrix} \rho_D(\xi_t) & 0 \\ 0 & \rho_S(\xi_t) \end{bmatrix} \begin{bmatrix} z_{D,t-1} \\ z_{S,t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{D,t} \\ \epsilon_{S,t} \end{bmatrix},$$

where $\epsilon_{D,t}$ is the innovation to a demand shock, $\epsilon_{S,t}$ is an innovation to the supply shock, and $\epsilon_{R,t}$ is a disturbance to the policy rule. All these structural shocks are i.i.d. and independent of one another. The standard deviations for these shocks are $\sigma_D(\xi_t)$, $\sigma_S(\xi_t)$, and $\sigma_R(\xi_t)$.

Lubik and Schorfheide (2004) estimate a constant-parameter version of this model for the two subsamples: 1960:I-1979:II and 1979:III-1997:IV. In our calibration we consider two regimes. The parameters in the first regime correspond to their estimates for the period 1979:III-1997:IV and the parameters in the second regime correspond to those for 1960:I-1979:II. The calibrated values are reported in Tables 1 and 2. The transition matrix is calculated by matching the average duration of the first regime to the length of the first subsample and by assuming that the second regime is absorbing to accommodate the belief that the pre-Volcker regime will never return:[6]

$$P = \begin{bmatrix} 0.9872 & 0 \\ 0.0128 & 1 \end{bmatrix}.$$

A simple calculation verifies that, if only one regime were allowed to exist (in the sense that a rational agent was certain that no other policy would ever be followed) the first regime would be indeterminate and the second would be determinate. When a rational agent forms expectations by taking account of regime changes, we need to know if there exists a unique MSV solution. To address this question, we stack the variables into the following 14-element vector, where $\boldsymbol{\iota}_{(\xi_t=1)}$ is the indicator function, $\boldsymbol{\iota} = 1$ if $\xi_t = i$ and 0 otherwise;

$$X_{1,t} = \left[\boldsymbol{\iota}_{(\xi_t=1)} \begin{pmatrix} x_t \\ E_t x_{t+1} \\ \pi_t \\ E_t \pi_{t+1} \\ R_t \\ z_{D,t} \\ z_{S,t} \end{pmatrix} \right], \quad X_{2,t} = \left[\boldsymbol{\iota}_{(\xi_t=2)} \begin{pmatrix} x_t \\ E_t x_{t+1} \\ \pi_t \\ E_t \pi_{t+1} \\ R_t \\ z_{D,t} \\ z_{S,t} \end{pmatrix} \right],$$

$$X_t = \begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix}.$$

The vector of the three i.i.d. fundamental shocks is

$$\epsilon_t \equiv \begin{bmatrix} \epsilon_{D,t} \\ \epsilon_{S,t} \\ \epsilon_{R,t} \end{bmatrix}.$$

Following the procedure illustrated in Section XIV, we write this regime-switching model as an expanded linear system:

$$AX_t = BX_{t-1} + \Psi u_t + \Pi \eta_t.$$

[6] One could also match the average duration of the second regime to the length of the second subsample, which give $p_{22} = 0.9865$. The unique solution exists for this situation as well and the findings presented in this section are unchanged.

In our computations we applied our method to this system, and in each parameterization that we report below, we obtained a unique MSV solution.

This kind of forward-looking model provides a natural laboratory to experiment with different scenarios in light of the recent debate on changes in policy or changes in shock variances. The estimates provided by Lubik and Schorfheide (2004) and reported in Tables 1 and 2 mix changes in coefficients related to monetary policy with changes in other parameters in the model since LS do not account for the effect of the probability of regime change on the current behavior. One variation in the structural parameter values is to let the coefficient on the inflation variable in the policy equation (44) change while holding all the other parameters fixed across the two regimes. Tables 3 and 4 report the parameter values corresponding to this scenario, in which all the other parameters take the average of the values in Tables 1 and 2 over the two regimes. We call this scenario "policy change only".

In a second scenario, "variance change only", we keep the value of the policy coefficient γ_1 at 2.19 for both regimes while letting the standard deviation σ_D in the first regime be five times larger than that in the second regime and keeping the value of σ_S at 0.3712 for both regimes.[7] The parameter values for this scenario are reported in Tables 5 and 6.

The last scenario we consider allows only the parameters in the private sector to change. We call it "private-sector change only". The idea is to study whether the persistence and volatility in inflation can be generated by changes in the private sector in a forward-looking model. We let the coefficient τ be 0.06137 in the first regime and 0.6137 in the second regime.[8] Tables 7 and 8 report the values of all the parameters for this scenario. Similar results can be achieved if one lets the value of κ in the first regime be much smaller than that in the second regime.

Using the method discussed in Section XIII, we obtain a unique MSV solution that characterizes each of our four scenarios. Figures 1-3 display simulated paths of the interest rate, inflation, and output gap under each of them. With the original estimates reported in Lubik and Schorfheide (2004), both inflation and the interest rate in the first regime display persistent and large fluctuations relative to their paths in the second regime. When we restrict changes to the policy coefficient γ_1 only, the results are similar. We find that smaller values of $\gamma_1(1)$, cause larger values for the persistence and volatility of R_t, π_t, and x_t.

When we allow only the variance to change, once again, all three variables have larger volatility in the first regime than the second (third panels in Figures 1-3). The degree of persistence and volatility in these variables in the first regime increases with

[7]Sims and Zha (2006) find that differences in the shock standard deviation across regimes can be on the scale of as high as $10 - 12$ times. One could also decrease the difference in σ_D and increase the difference in σ_S or experiment with different combinations. Our result that changes in variances matter a great deal will hold.

[8]It is not unlikely that the maximum likelihood estimate of τ is close to zero or even has a wrong sign. The Bayesian prior that puts almost zero probability on the region of τ near zero, as typically done in the existing literature, is unreasonable. See Fukac and Pagan (2006).

persistence of the shock $z_{D,t}$ or $z_{S,t}$ and with the size of shock variance $\sigma_{D,t}$ or σ_{S_t}. In our final scenario, we show that even if there is no change in the structural variances, inflation and the interest rate can have much larger fluctuations in the first regime than in the second regime when the parameters of the private sector equations are allowed to change across regimes. The bottom panels of Figures 1 - 3 illustrate this case.

These examples teach us that the sharply different dynamics in inflation, the interest rate, and output observed before and after 1980 could potentially be attributed to different sources. The methods we have developed here give researchers the tools to address this and other issues in a regime-switching forward looking model in which a rational agent takes into account the probability of regime change when forming his expectations.

XVI. Conclusion

In related work, Farmer et. al., (2006), we showed that indeterminacy is pervasive in a simple example of an MSRE model with no lagged state variables. Since there is no reason to think that the purely forward looking example is special in any essential way, there is an urgent need to refine the solution concept used in MSRE models, beyond that of a bounded sequence that satisfies a given functional equation. Arguably, the MSV solution, introduced by McCallum, is the most natural candidate. MSV solutions are typically stable under real time learning and, as we demonstrated in Section XV, they are rich enough to explain heteroskedasticity across policy regimes within a model in which all of the private sector equations are invariant across regimes.

Our paper contains two main parts. First, we provided a method to write a variable parameter rational expectations model in expanded form with state invariant parameters and we found conditions under which a solution to the expanded model is also a solution to the original one. We showed how to compute an MSV solution to the expanded model and to check to see if this solution is unique and bounded. In the second part of the paper, we applied our methods to the New Keynesian model and showed that the MSV solution provides a natural way to compare alternative parsimonious explanations of observed US data.

TABLE 1. Model coefficients (original)

	Structural Equations				
Parameter	τ	κ	β	γ_1	γ_2
First regime	0.69	0.77	0.997	0.77	0.17
Second regime	0.54	0.58	0.993	2.19	0.30

TABLE 2. Shock variances (original)

	Shock Processes					
Parameter	ρ_D	ρ_S	ρ_R	σ_D	σ_S	σ_R
First regime	0.68	0.82	0.60	0.27	0.87	0.23
Second regime	0.83	0.85	0.84	0.18	0.37	0.18

TABLE 3. Model coefficients (policy change only)

	Structural Equations				
Parameter	τ	κ	β	γ_1	γ_2
First regime	0.6137	0.6750	0.9949	0.77	0.235
Second regime	0.6137	0.6750	0.9949	2.19	0.235

TABLE 4. Shock variances (policy change only)

	Shock Processes					
Parameter	ρ_D	ρ_S	ρ_R	σ_D	σ_S	σ_R
First regime	0.755	0.835	0.72	0.225	0.6206	0.205
Second regime	0.755	0.835	0.72	0.225	0.6206	0.205

TABLE 5. Model coefficients (variance change only)

	Structural Equations				
Parameter	τ	κ	β	γ_1	γ_2
First regime	0.6137	0.6750	0.9949	2.19	0.235
Second regime	0.6137	0.6750	0.9949	2.19	0.235

TABLE 6. Shock variances (variance change only)

	Shock Processes					
Parameter	ρ_D	ρ_S	ρ_R	σ_D	σ_S	σ_R
First regime	0.755	0.835	0.72	0.225	0.3712	0.205
Second regime	0.755	0.835	0.72	1.125	0.3712	0.205

TABLE 7. Model coefficients (private sector change only)

	Structural Equations				
Parameter	τ	κ	β	γ_1	γ_2
First regime	0.0614	0.6750	0.9949	2.19	0.235
Second regime	0.6137	0.6750	0.9949	2.19	0.235

TABLE 8. Shock variances (private sector change only)

	Shock Processes					
Parameter	ρ_D	ρ_S	ρ_R	σ_D	σ_S	σ_R
First regime	0.755	0.835	0.72	0.225	0.6206	0.205
Second regime	0.755	0.835	0.72	0.225	0.6206	0.205

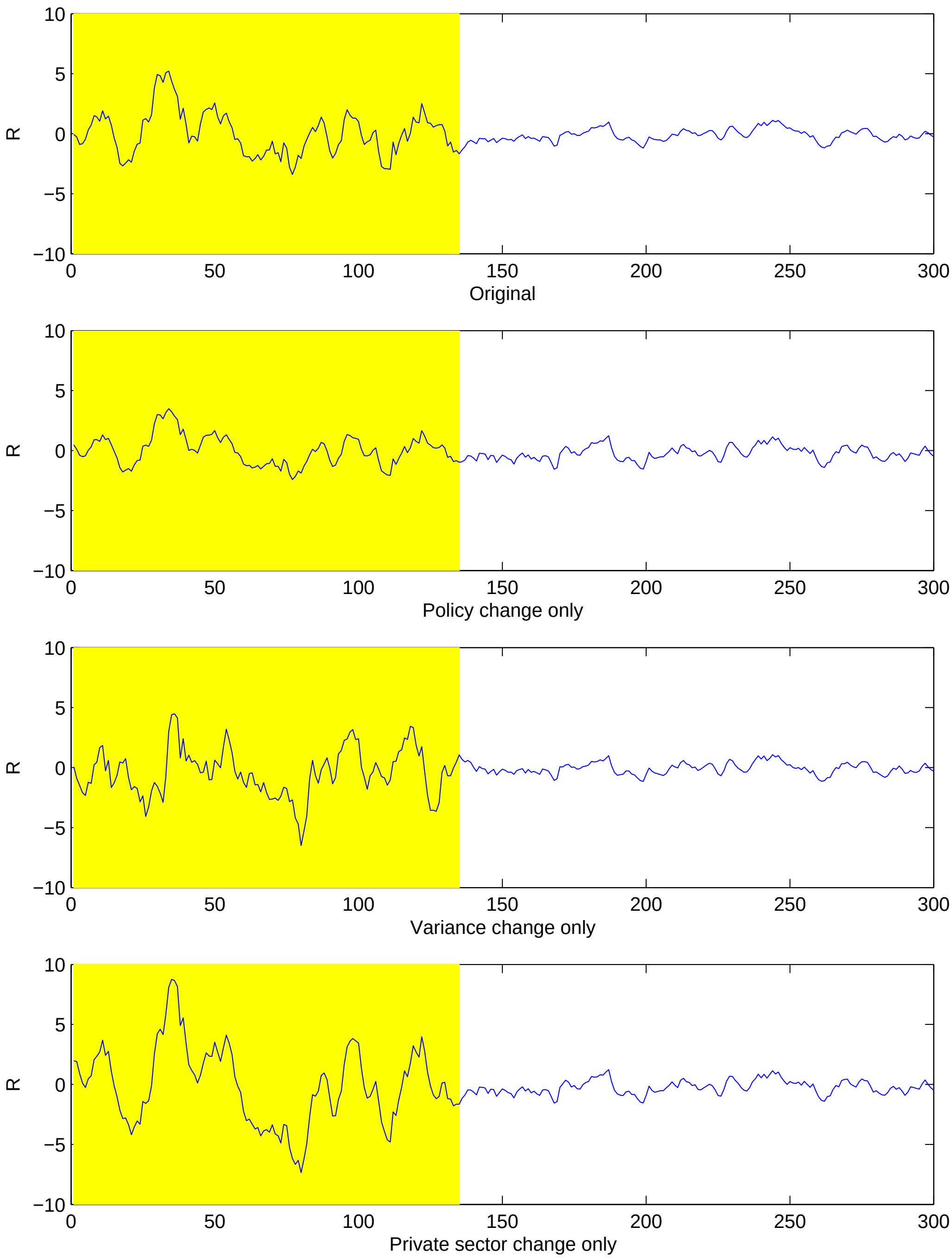

FIGURE 1. Simulated interest rate paths from our regime-switching forward looking model. The shaded area represents the first regime.

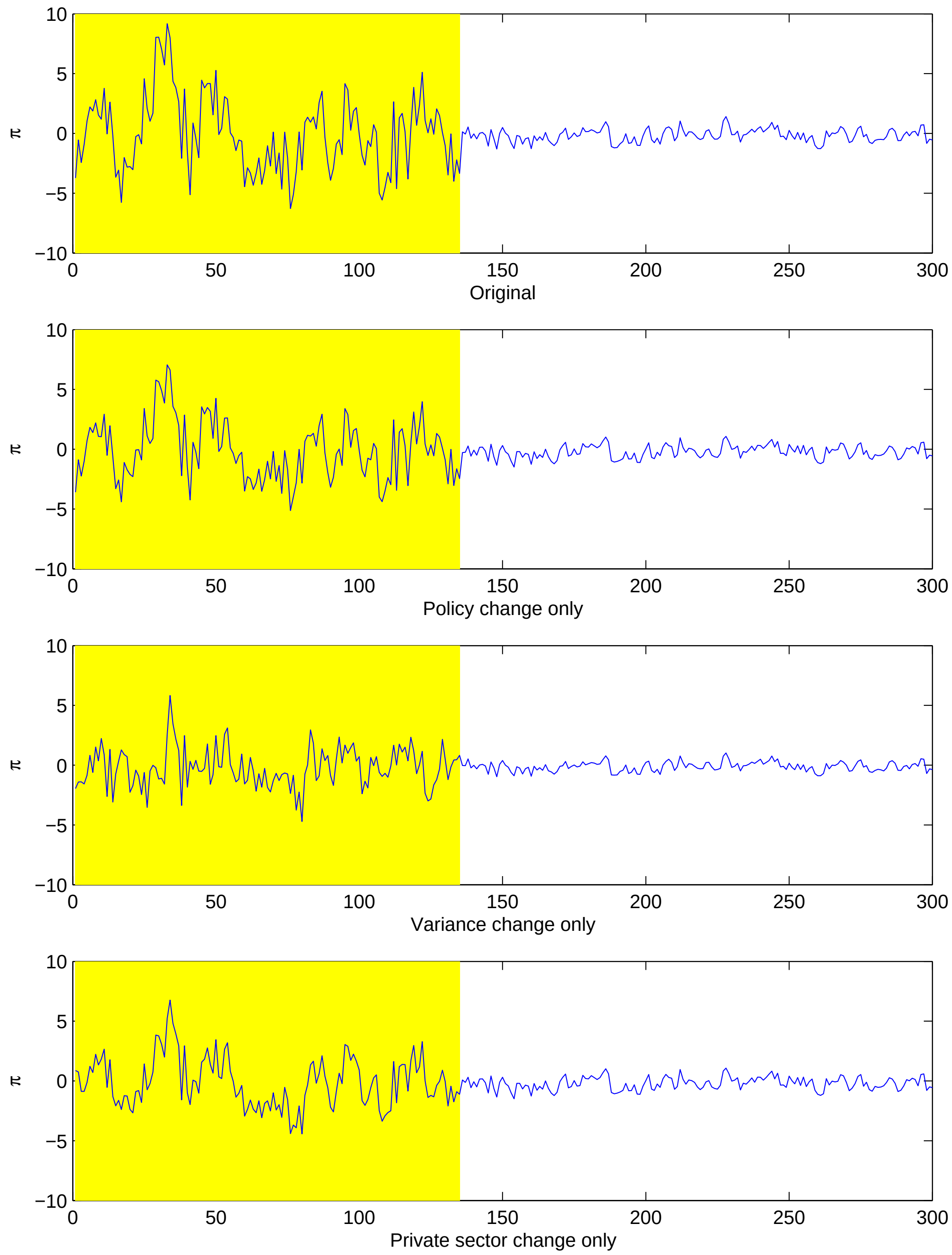

FIGURE 2. Simulated inflation paths from our regime-switching forward looking model. The shaded area represents the first regime.

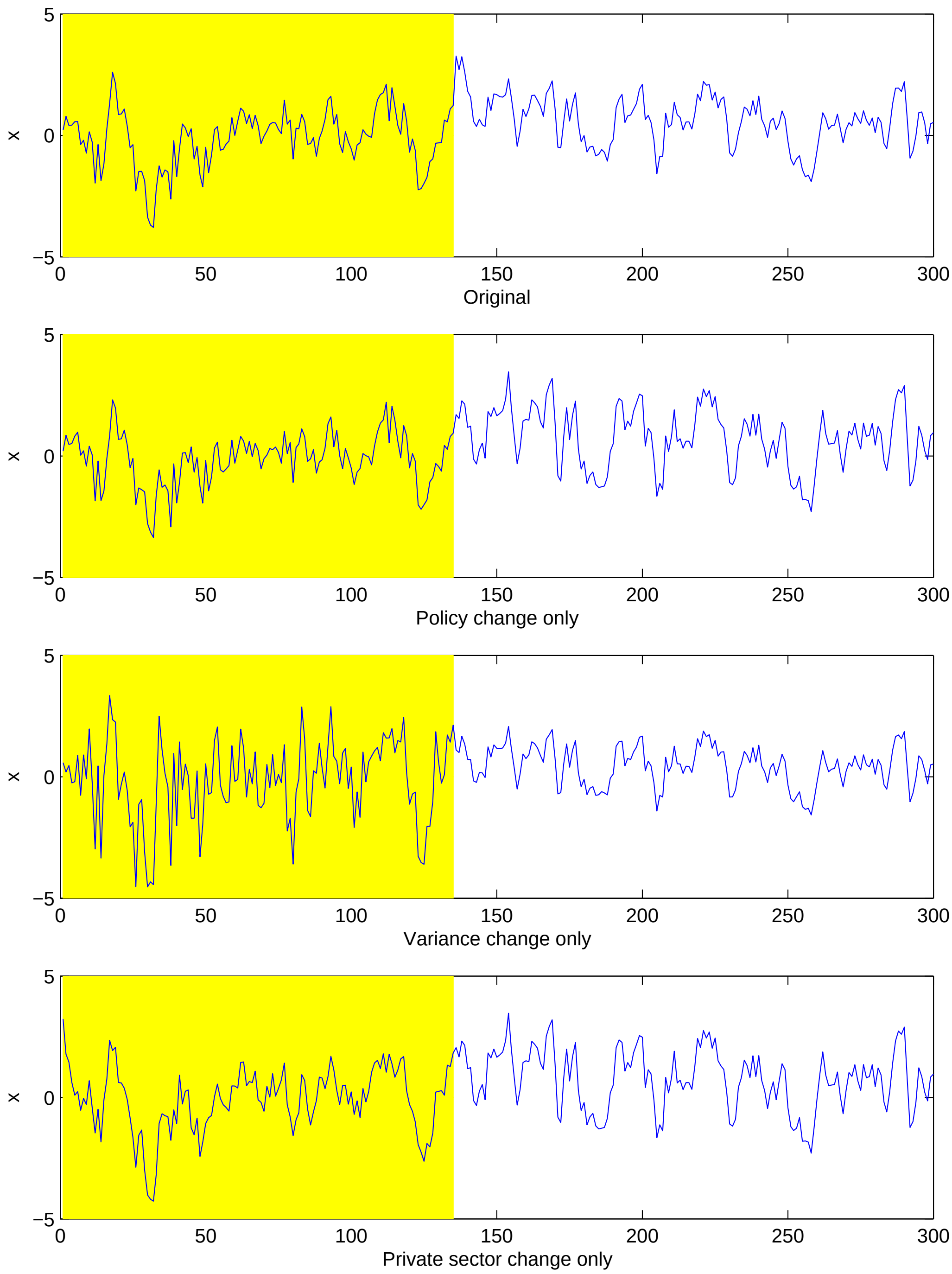

FIGURE 3. Simulated output gap paths from our regime-switching forward looking model. The shaded area represents the first regime.

Appendix A. Proof of Theorems

A.1. **Proof of Theorem 1.** Let $\{x_t, \eta_t\}_{t=1}^{\infty}$ be a MSV solution of Equation (8). Denote the span of this solution, conditional on $\xi_t = j$, by $\hat{V}_j$ and let V_j be any $n \times (n-\ell)$ matrix whose columns span $\hat{V}_j$. Applying the $E_{t-1}[\cdot|\xi_t = j]$ operator to Equation (8) gives

$$A(j)E_{t-1}[x_t|\xi_t = j] = B(j)x_{t-1} + \Pi E_{t-1}[\eta_t|\xi_t = j]. \tag{A1}$$

This implies that every element of $B(j)V_i$ is a linear combination of the columns of the matrix $\begin{bmatrix} A(j)V_j & \Pi \end{bmatrix}$. Thus there exist $(n-\ell)\times(n-\ell)$ matrices $F_{i,j}^1$ and $\ell\times(n-\ell)$ matrices $F_{i,j}^2$ such that

$$\begin{bmatrix} A(j)V_j & \Pi \end{bmatrix} \begin{bmatrix} F_{i,j}^1 \\ F_{i,j}^2 \end{bmatrix} = B(j)V_i.$$

Furthermore, since

$$\begin{aligned} \sum_{j=1}^{h} p_{\xi_{t-1},j} A(j)E_{t-1}[x_t|\xi_t = j] &= \sum_{j=1}^{h} p_{\xi_{t-1},j}\left(B(j)x_{t-1} + \Pi E_{t-1}[\eta_t|\xi_t = j]\right) \\ &= \sum_{j=1}^{h} p_{\xi_{t-1},j} B(j)x_{t-1} \end{aligned}$$

and Π is of full column rank, we can choose the $F_{i,j}^1$ and $F_{i,j}^2$ so that

$$\sum_{j=1}^{h} p_{i,j} F_{i,j}^2 = 0_{\ell,n-\ell}.$$

Subtracting Equation (A1) from Equation (8) gives

$$A(j)\left(x_t - E_{t-1}[x_t|\xi_t = j]\right) = \Psi(j)\varepsilon_t + \Pi\left(\eta_t - E_{t-1}[\eta_t|\xi_t = j]\right).$$

This implies that there exist $(n-\ell)\times k$ matrices G_j^1 and $\ell \times k$ matrices G_j^2 such that

$$\begin{bmatrix} A(j)V_j & \Pi \end{bmatrix} \begin{bmatrix} G_j^1 \\ G_j^2 \end{bmatrix} = \Psi(j).$$

Let V_j^* denote the generalized inverse of V_j and define

$$\begin{aligned} \hat{x}_t &= V_{\xi_t} F_{\xi_{t-1},\xi_t}^1 V_{\xi_{t-1}}^* \hat{x}_{t-1} + V_{\xi_t} G_{\xi_t}^1 \varepsilon_{t-1}, \\ \hat{\eta}_t &= -\left(F_{\xi_{t-1},\xi_t}^2 V_{\xi_{t-1}}^* \hat{x}_{t-1} + G_{\xi_t}^2 \varepsilon_{t-1}\right). \end{aligned}$$

This will also be a solution of Equation (8) whose span, conditional on $\xi_t = j$, is $\hat{V}_j$. Since $\{x_t, \eta_t\}_{t=1}^{\infty}$ is a MSV solution, it must be the case that $\hat{x}_t = x_t$ and $\hat{\eta}_t = \eta_t$.

Finally, $\begin{bmatrix} A(j)V_j & \Pi \end{bmatrix}$ must be invertible because otherwise the G_j^1 and G_j^2 would not be unique. So, define

$$\begin{bmatrix} F_j^1 \\ F_j^2 \end{bmatrix} = \begin{bmatrix} A(j)V_j & \Pi \end{bmatrix}^{-1} B(j).$$

It is easy to see that $F_j^1 V_i = F_{i,j}^1 = F_{i,j}^1 V_i^* V_i$ and $F_j^2 V_i = F_{i,j}^2 = F_{i,j}^2 V_i^* V_i$. Thus

$$\left(\sum_{j=1}^{h} p_{i,j} F_j^2\right) V_i = 0_{\ell,n-\ell},$$

and

$$\begin{aligned} x_t &= V_{\xi_t} F_{\xi_t}^1 x_{t-1} + V_{\xi_t} G_{\xi_t}^1 \varepsilon_{t-1}, \\ \eta_t &= -\left(F_{\xi_t}^2 x_{t-1} + G_{\xi_t}^2 \varepsilon_{t-1}\right). \end{aligned}$$

A.2. **Newton's Method.** Newton's method is an iterative technique for finding roots. It has excellent local convergence properties and, while global convergence is not guaranteed, in practice it global convergence properties are good. If $\left(X_1^{(i)}, \cdots, X_h^{(i)}\right)$ is the i^{th} iteration, then the $(i+1)^{\text{th}}$ iteration is given by

$$\begin{aligned} vec\left(X_1^{(i+1)}, \cdots, X_h^{(i+1)}\right) &= vec\left(X_1^{(i)}, \cdots, X_h^{(i)}\right) \\ &\quad - f'\left(X_1^{(i)}, \cdots, X_h^{(i)}\right)^{-1} vec\left(f\left(X_1^{(i+1)}, \cdots, X_h^{(i+1)}\right)\right), \end{aligned}$$

where

$$f'\left(X_1, \cdots, X_h\right) = \begin{bmatrix} f_1'\left(X_1, \cdots, X_h\right) \\ \vdots \\ f_h'\left(X_1, \cdots, X_h\right) \end{bmatrix},$$

and

$$f_i'\left(X_1, \cdots, X_h\right) = \left(\frac{\partial f_i}{\partial X_1}\left(X_1, \cdots, X_h\right), \cdots, \frac{\partial f_i}{\partial X_h}\left(X_1, \cdots, X_h\right)\right).$$

A straight forward calculation shows

$$\frac{\partial f_i}{\partial X_j}\left(X_1, \cdots, X_h\right) = p_{i,j}\left(\begin{bmatrix} I_{n-\ell} & 0_{n-\ell,\ell} \end{bmatrix} B(j) A(j)^{-1} \begin{bmatrix} I_{n-\ell} \\ -X_i \end{bmatrix}\right)' \otimes I_\ell$$

if $i \neq j$ and

$$\begin{aligned} \frac{\partial f_i}{\partial X_i}\left(X_1, \cdots, X_h\right) &= p_{i,i}\left(\begin{bmatrix} I_{n-\ell} & 0_{n-\ell,\ell} \end{bmatrix} B(j) A(j)^{-1} \begin{bmatrix} I_{n-\ell} \\ -X_i \end{bmatrix}\right)' \otimes I_\ell \\ &\quad + I_{n-\ell} \otimes \left(\sum_{k=1}^{h} p_{i,k} \begin{bmatrix} X_k & I_\ell \end{bmatrix} B(k)\right) A(i)^{-1} \begin{bmatrix} 0_{n-\ell,\ell} \\ -I_\ell \end{bmatrix} \end{aligned}$$

if $i = j$. To be completed is the case that A_j is singular.

A.3. **Proof of Theorem 2.** Because $\{x_t, \eta_t\}_{t=1}^{\infty}$ satisfies Equation (23), $\{X_t, \eta_t\}_{t=1}^{\infty}$ will satisfy the first two blocks of Equation (23). Because the row space of ϕ_i is contained in the row space of χ_i, $\{X_t, \eta_t\}_{t=1}^{\infty}$ will satisfy the last block of Equation (23). Because of the Kronecker product structure of X_t, $E_t\left[X_{t+s}\right]$ will be bounded if and only if $E_t\left[x_{t+s}\right]$ is bounded. Finally, $E_{t-1}\left[\eta_t\right]$ because $\{x_t, \eta_t\}_{t=1}^{\infty}$ is a solution of Equation (23). Thus $\{X_t, \eta_t\}_{t=1}^{\infty}$ is a solution of Equation (23).

A.4. **Proof of Theorem 3.** The crux of the proof will be to show that any solution $\{X_t, \eta_t\}_{t=1}^{\infty}$ of Equation (23) with $X_0 = e_{\xi_0} \otimes x_0$ must be of the form $X_t = e_{\xi_t} \otimes x_t$ for all t. Because Equation is a linear system, it follows easily from Sims (2001) that any solution of Equation (23) must satisfy $Z_u X_t = 0$ for all $0 \leq t$. This together with the facts that $(e_i' \otimes \phi_i) X_t = 0$ and that the row space of z_i is equal to the row space of ϕ_i for all $0 < t$ and $2 \leq i \leq m$, imply that $(e_i' \otimes z_i) X_t = 0$ for all $0 < t$ and $1 \leq i \leq m$.

We show that $X_t = e_{\xi_t} \otimes x_t$, by induction on t. By assumption, $X_0 = e_{\xi_0} \otimes x_0$. Now assume that $X_{t-1} = e_{\xi_{t-1}} \otimes x_{t-1}$. Let $X_t' = [x_1', \cdots, x_h']$, where x_i is a n dimensional column vector. From Equation (23), it is easy to see that

$$\begin{bmatrix} a_1(i) \\ z_i \end{bmatrix} x_i = 0$$

for $i \neq \xi_t$. Since $\begin{bmatrix} a_1'(i) & z_i' \end{bmatrix}$ has rank n, this implies that $x_i = 0$ for $i \neq \xi_t$. Thus X_t has the required Kronecker product structure.

All that remains to be shown is that $\{x_t, \eta_t\}_{t=1}^{\infty}$ is a solution of Equation (20). The first two blocks of equations in (23) together with the definition of u_t and S_i and the Kronecker product structure of X_t, imply that x_t and η_t satisfy Equation (20). Because $\{X_t, \eta_t\}_{t=1}^{\infty}$ is a solution of Equation (23), $E_{t-1}[\eta_t] = 0$. Given the Kronecker product structure of X_t, it is clear that $E_t[X_{t+s}]$ will be bounded if and only if $E_t[x_{t+s}]$ is bounded. Thus all the conditions for $\{x_t, \eta_t\}_{t=1}^{\infty}$ to be a solution of Equation (20) are satisfied.

A.5. **Proof of Theorem 4.** From Sims (2001), a solution of Equation (23) will exist if and only if there exists a matrix Λ such that $Z_2(\Psi + \Pi\Lambda) = 0$. So, if $Z_2(\Psi + \Pi\Lambda) = 0$, then a solution of Equation exists and from Theorem (3), this solution is of the form $X_t = \mathbf{e}_{\xi_t} \otimes x_t$ where x_t is a solution of Equation (20). From Lubik and Schorfheide (2003)[page 277], we know that any solution of linear system given by Equation (23) will be of the form (X_t, η_t) where

$$X_t = G_1 X_{t-1} + G_2 u_t + G_3 \gamma_t$$

The term $G_3\gamma_t$ will be present only if the solution of Equation (23) is not unique. In this case, the random variable γ_t can be any mean zero process. Suppose that the solution were not unique. Since γ_t can be any mean zero process, it can be taken to be identically zero. In this case, X_t would be equal to $G_1 X_{t-1} + G_2 u_t$. Since X_t must have a Kronecker product form, this implies that $G_1 X_{t-1} + G_2 u_t$ must have a Kronecker product form. This, in turn, implies that $G_3\gamma_t$ must have a Kronecker product form for all mean zero process γ_t. But this can happen only if G_3 is zero. This implies that X_t is a unique solution of Equation (23). If x_t is any solution of Equation (20) satisfying $\phi_{\xi_t} x_t = 0$ for all $t > 0$, then $X_t = \mathbf{e}_{\xi_t} \otimes x_t$ will be a solution of Equation 23. Since these solutions are unique, so must be solutions of Equation (20) satisfying $\phi_{\xi_t} x_t = 0$ for all $t > 0$.

A.6. **Proof of Theorem 5.** To be added.

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