Minimal State Variable Solutions to Markov-Switching Rational Expectations Models

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Modeling Regime Shifts

- Structural changes in consumers and firms' behavior, in policy, or in volatility are often modeled by discrete changes in the coefficients of economic models.
- Backward looking Markov-switching models have been popular tools since Hamilton (1989), but
- In an economy where past structural changes are observed and future changes are likely, rational agents will form a probability distribution over possible structural changes in the future when forming expectations (Lucas 1976; Cooley, LeRoy, and Raymon 1984; Sims 1992).
- The Markov-switching approach has been extended to a rational expectations framework (Leeper and Zha 2003; Svensson and Williams 2005; Davig and Leeper 2006, 2007; Farmer, Waggoner and Zha 2008; Liu, Waggoner and Zha 2008), but better solution techniques are needed.

Our Contributions

1. We characterize the general solutions of linear rational expectations model with exogenous Markov-switching of the parameters.
2. We extend the notion of the minimum state variable (MSV) solution (MacCallum (1983)) to the Markov-switching case.
3. We derive an easy to use and efficient algorithm for obtaining MSV solutions.

General Model

- The models we study are represented by the equation,

$$A_{1,s_t} \begin{bmatrix} z_t \\ y_t \end{bmatrix} = A_{2,s_t} E_t y_{t+1} + B_{1,s_t} \begin{bmatrix} z_{t-1} \\ y_{t-1} \end{bmatrix} + \Psi_{1,s_t} \varepsilon_t \qquad (1)$$

- The process s_t is Markov, taking values between 1 and h with transition matrix $P = [p_{ij}]$. The element p_{ij} represents the probability that $s_t = j$ given $s_{t-1} = i$.
- y_t is an n-dimensional vector of endogenous random variables
- z_t is an m-dimensional vector of predetermined variables
- ε_t is an r-dimensional vector of exogenous shocks that are independent of the Markov process s_t
- $A_{1,j}$, $B_{1,j}$ - $(m+n) \times (m+n)$
  $A_{2,j}$ - $(m+n) \times n$
  $\Psi_{1,j}$ - $(m+n) \times r$

General Model (continued)

These can be written in the expectational error form

$$A_{s_t} x_t = B_{s_t} x_{t-1} + \Psi_{s_t} \varepsilon_t + \Pi \eta_t, \tag{2}$$

where

$$x_t = \begin{bmatrix} z_t \\ y_t \\ E_t y_{t+1} \end{bmatrix} \quad A_j = \begin{bmatrix} A_{1,j} & -A_{2,j} \\ \begin{bmatrix} 0_{n,m} & I_n \end{bmatrix} & 0_{n,n} \end{bmatrix}$$

$$B_j = \begin{bmatrix} B_{1,j} & 0_{n+m,n} \\ 0_{n,m+n} & I_n \end{bmatrix} \quad \Psi_j = \begin{bmatrix} \Psi_{1,j} \\ 0_{n,r} \end{bmatrix} \quad \Pi = \begin{bmatrix} 0_{n+k,n} \\ I_n \end{bmatrix}$$

MSV Solutions

- **Where does the solution live - Conditional Span**
  The span of the solution, conditional on $s_t = i$, is the smallest linear space that contains the support of $x_t 1_{\{s_t = i\}}$ for every t, where $1_{\{s_t = i\}}$ is the indicator function that is 1 if $s_t = i$.

- If the conditional span is smaller than the entire space, then there is a linear relation among z_t, y_t, and $E_t y_{t+1}$.

- The conditional span of a MSV solution is small enough so that $E_t y_{t+1}$ is uniquely determined given z_t and y_t, but not so small that z_t and y_t are constrained.

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- This implies that the dimension of the conditional span is $m + n$.

MSV Solutions (continued)

Theorem

The processes x_t and η_t are a MSV solution of (2) if and only if they are of the form

$$\begin{aligned} x_t &= V_{s_t}\left(F^1_{s_t} x_{t-1} + G^1_{s_t}\varepsilon_t\right) \\ \eta_t &= -\left(F^2_{s_t} x_{t-1} + G^2_{s_t}\varepsilon_t\right) \end{aligned}$$

where the column space of V_i is the span of the solution conditional on $s_t = i$, $\begin{bmatrix} A_i V_i & \Pi \end{bmatrix}$ is invertible,

$$\begin{bmatrix} A_i V_i & \Pi \end{bmatrix} \begin{bmatrix} F^1_i \\ F^2_i \end{bmatrix} = B_i, \quad \begin{bmatrix} A_i V_i & \Pi \end{bmatrix} \begin{bmatrix} G^1_i \\ G^2_i \end{bmatrix} = \Phi_i,$$

and

$$\sum_{j=1}^{h} p_{i,j} F^2_j V_i = 0 \text{ for } 1 \le i \le h. \tag{3}$$

Finding MSV solutions

- Given the representation of MSV solutions, we could use the method of undetermined coefficients to iteratively find MSV solutions. This is essentially the method devised by Svensson and Williams (2005).
- Use an iteratively defined constant parameter stacked system to find both the conditional span and solution using Sims (2001). This technique was proposed in an earlier version of this paper.
- Solve equation (3) to find the conditional span of an MSV solution.

Newton's Method

- Since $\begin{bmatrix} A_i V_i & \Pi \end{bmatrix}$ is invertible and V_i is only defined up to right multiplication by an invertible matrix, we may assume that $A_i V_i$ is of the form

$$A_i V_i = \begin{bmatrix} I_{m+n} \\ X_i \end{bmatrix}$$

where X_i is a $n \times (m+n)$ matrix of free parameters.

- If A_i is invertible, then (3) becomes

$$\sum_{j=1}^{h} p_{i,j} \begin{bmatrix} X_j & I_n \end{bmatrix} B(j) A(i)^{-1} \begin{bmatrix} I_{m+n} \\ -X_i \end{bmatrix} = 0_{n,m+n}.$$

- This is a quadratic system of $hn(m+n)$ equations in $hn(m+n)$ unknowns and can be quickly solved using your favorite root finding algorithm, such as Newton's method.

Stable Solutions

- When is a process of the form

$$x_t = V_{s_t} F^1_{s_t} x_{t-1} + V_{s_t} G^1_{s_t} \varepsilon_t$$

stable?

- It follows from Costa, Fragoso, and Marques (2004) that this system will be mean square stable if and only if all the eigenvalues of

$$\begin{bmatrix} p_{1,1} V_1 F^1_1 \otimes V_1 F^1_1 & \cdots & p_{h,1} V_h F^h_1 \otimes V_h F^1_h \\ \vdots & \ddots & \vdots \\ p_{1,h} V_1 F^1_1 \otimes V_1 F^1_1 & \cdots & p_{h,h} V_h F^h_1 \otimes V_h F^1_h \end{bmatrix}$$

are inside the unit circle.

- In the case that there are multiple MSV solutions, choosing the one with smaller eigenvalues will result in a solution with smaller variance.

Example

Consider the regime-switching policy model, based on Lubik and Schorfheide (2004).

$$
\begin{aligned}
x_t &= E_t x_{t+1} - \tau(s_t)(R_t - E_t \pi_{t+1}) + z_{D,t}, \\
\pi_t &= \beta(s_t) E_t \pi_{t+1} + \kappa(s_t) x_t + z_{S,t}, \\
R_t &= \rho_R(s_t) R_{t-1} + (1 - \rho_R(s_t)) \left[\gamma_1(s_t)\pi_t + \gamma_2(s_t) x_t\right] + \epsilon_{R,t},
\end{aligned}
$$

where x_t is the output gap t, π_t is the inflation rate and R_t is the nominal interest rate with both π_t and R_t are measured in terms of deviations from the regime-dependent state. The shocks to the consumer and firm's sectors evolve as

$$
\begin{bmatrix} z_{D,t} \\ z_{S,t} \end{bmatrix} = \begin{bmatrix} \rho_D(s_t) & 0 \\ 0 & \rho_S(s_t) \end{bmatrix} \begin{bmatrix} z_{D,t-1} \\ z_{S,t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{D,t} \\ \epsilon_{S,t} \end{bmatrix},
$$

Example (continued)

Lubik and Schorfheide estimated the model assuming that the agents behaved as if each regime were to last indefinitely. The parameter estimates are given below:

	Structural Equations				
Parameter	τ	κ	β	γ_1	γ_2
First regime	0.69	0.77	0.997	0.77	0.17
Second regime	0.54	0.58	0.993	2.19	0.30

	Shock Processes					
Parameter	ρ_D	ρ_S	ρ_R	σ_D	σ_S	σ_R
First regime	0.68	0.82	0.60	0.27	0.87	0.23
Second regime	0.83	0.85	0.84	0.18	0.37	0.18

Example (continued)

- Considering each regime in isolation, the first regime would be indeterminate while the second regime is determinate.
- Given the above calibration, on could compute the MSV solution, which would be different from the solutions obtained by considering each regime in isolation.
- Given any parameterization, one can find MSV solutions of the model (assuming that any exist), and could use the solution to approximate the likelihood.
- These techniques could thus be used to estimate the model taking into account that agents know that the regimes could switch.

Conclusions

- We extend the notion of the MSV solution (MacCallum (1983)) to Markov-switching rational expectations models.
- We characterize both the general and MSV solutions of linear rational expectations model with Markov-switching parameters.
- We derive an easy to use and efficient algorithm for obtaining MSV solutions.