Minimal State Variable Solutions to Markov-Switching Rational Expectations Models

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Modeling Regime Shifts

- Structural changes in consumers and firms’ behavior, in policy, or in volatility are often modeled by discrete changes in the coefficients of economic models.

- Backward looking Markov-switching models have been popular tools since Hamilton (1989), but

- In an economy where past structural changes are observed and future changes are likely, rational agents will form a probability distribution over possible structural changes in the future when forming expectations (Lucas 1976; Cooley, LeRoy, and Raymon 1984; Sims 1992).

- The Markov-switching approach has been extended to a rational expectations framework (Leeper and Zha 2003; Svensson and Williams 2005; Davig and Leeper 2006, 2007; Farmer, Waggoner and Zha 2008; Liu, Waggoner and Zha 2008), but better solution techniques are needed.
Our Contributions

1. We characterize the general solutions of linear rational expectations model with exogenous Markov-switching of the parameters.
2. We extend the notion of the minimum state variable (MSV) solution (MacCallum (1983)) to the Markov-switching case.
3. We derive an easy to use and efficient algorithm for obtaining MSV solutions.
General Model

- The models we study are represented by the equation,

\[
A_{1,s_t} \begin{bmatrix} z_t \\ y_t \end{bmatrix} = A_{2,s_t} E_t y_{t+1} + B_{1,s_t} \begin{bmatrix} z_{t-1} \\ y_{t-1} \end{bmatrix} + \Psi_{1,s_t} \epsilon_t
\] (1)

- The process \( s_t \) is Markov, taking values between 1 and \( h \) with transition matrix \( P = [p_{ij}] \). The element \( p_{ij} \) represents the probability that \( s_t = j \) given \( s_{t-1} = i \).

- \( y_t \) is an \( n \)-dimensional vector of endogenous random variables

- \( z_t \) is an \( m \)-dimensional vector of predetermined variables

- \( \epsilon_t \) is an \( r \)-dimensional vector of exogenous shocks that are independent of the Markov process \( s_t \)

- \( A_{1,j}, B_{1,j} - (m + n) \times (m + n) \)
  
- \( A_{2,j} - (m + n) \times n \)
  
- \( \Psi_{1,j} - (m + n) \times r \)
General Model (continued)

These can be written in the expectational error form

\[ A_s t x_t = B_s t x_{t-1} + \psi_s t \varepsilon_t + \Pi \eta_t, \]  

(2)

where

\[
\begin{align*}
  x_t &= \begin{bmatrix} z_t \\ y_t \\ E_t y_{t+1} \end{bmatrix} \\
  A_j &= \begin{bmatrix} A_{1,j} & -A_{2,j} \\ [0_{n,m} & I_n] & 0_{n,n} \end{bmatrix} \\
  B_j &= \begin{bmatrix} B_{1,j} & 0_{n+m,n} \\ 0_{n,m+n} & I_n \end{bmatrix} \\
  \psi_j &= \begin{bmatrix} \psi_{1,j} \\ 0_{n,r} \end{bmatrix} \\
  \Pi &= \begin{bmatrix} 0_{n+k,n} \\ I_n \end{bmatrix}
\end{align*}
\]
MSV Solutions

- **Where does the solution live - Conditional Span**
  The span of the solution, conditional on $s_t = i$, is the smallest linear space that contains the support of $x_t 1_{\{s_t=i\}}$ for every $t$, where $1_{\{s_t=i\}}$ is the indicator function that is 1 if $s_t = i$.

- If the conditional span is smaller than the entire space, then there is a linear relation among $z_t$, $y_t$, and $E_t y_{t+1}$.

- The conditional span of a MSV solution is small enough so that $E_t y_{t+1}$ is uniquely determined given $z_t$ and $y_t$, but not so small that $z_t$ and $y_t$ are constrained.
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- This implies that the dimension of the conditional span is $m + n$. 
Theorem

The processes $x_t$ and $\eta_t$ are a MSV solution of (2) if and only if they are of the form

\[
\begin{align*}
  x_t & = V_s t \left( F_{s t}^1 x_{t-1} + G_{s t}^1 \varepsilon_t \right) \\
  \eta_t & = - \left( F_{s t}^2 x_{t-1} + G_{s t}^2 \varepsilon_t \right)
\end{align*}
\]

where the column space of $V_i$ is the span of the solution conditional on $s_t = i$, $[A_i V_i \ \Pi]$ is invertible,

\[
\begin{align*}
  [A_i V_i \ \Pi] \begin{bmatrix} F_i^1 \\ F_i^2 \end{bmatrix} & = B_i, \\
  [A_i V_i \ \Pi] \begin{bmatrix} G_i^1 \\ G_i^2 \end{bmatrix} & = \Phi_i,
\end{align*}
\]

and

\[
\sum_{j=1}^{h} p_{i,j} F_j^2 V_i = 0 \text{ for } 1 \leq i \leq h.
\]
Finding MSV solutions

- Given the representation of MSV solutions, we could use the method of undetermined coefficients to iteratively find MSV solutions. This is essentially the method devised by Svensson and Williams (2005).
- Use an iteratively defined constant parameter stacked system to find both the conditional span and solution using Sims (2001). This technique was proposed in an earlier version of this paper.
- Solve equation (3) to find the conditional span of an MSV solution.
Newton’s Method

• Since $[A_i; V_i \ \Pi]$ is invertible and $V_i$ is only defined up to right multiplication by an invertible matrix, we may assume that $A_i V_i$ is of the form

$$A_i V_i = \begin{bmatrix} I_{m+n} \\ X_i \end{bmatrix}$$

where $X_i$ is a $n \times (m + n)$ matrix of free parameters.

• If $A_i$ is invertible, then (3) becomes

$$\sum_{j=1}^{h} p_{i,j} \begin{bmatrix} X_j & I_n \end{bmatrix} B(j) A(i)^{-1} \begin{bmatrix} I_{m+n} \\ -X_i \end{bmatrix} = 0_{n,m+n}.$$  

• This is a quadratic system of $hn(m + n)$ equations in $hn(m + n)$ unknowns and can be quickly solved using your favorite root finding algorithm, such as Newton’s method.
Stable Solutions

• When is a process of the form

\[ x_t = V_{st} F_{st}^1 x_{t-1} + V_{st} G_{st}^1 \epsilon_t \]

stable?

• It follows from Costa, Fragoso, and Marques (2004) that this system will be mean square stable if and only if all the eigenvalues of

\[
\begin{bmatrix}
p_{1,1} V_1 F_1^1 \otimes V_1 F_1^1 & \cdots & p_{h,1} V_h F_h^h \otimes V_h F_h^1 \\
\vdots & \ddots & \vdots \\
p_{1,h} V_1 F_1^1 \otimes V_1 F_1^1 & \cdots & p_{h,h} V_h F_h^h \otimes V_h F_h^1
\end{bmatrix}
\]

are inside the unit circle.

• In the case that there are multiple MSV solutions, choosing the one with smaller eigenvalues will result in a solution with smaller variance.
Example

Consider the regime-switching policy model, based on Lubik and Schorfheide (2004).

\[ x_t = E_t x_{t+1} - \tau(s_t)(R_t - E_t \pi_{t+1}) + z_{D,t}, \]
\[ \pi_t = \beta(s_t)E_t \pi_{t+1} + \kappa(s_t)x_t + z_{S,t}, \]
\[ R_t = \rho_R(s_t)R_{t-1} + (1 - \rho_R(s_t)) \left[ \gamma_1(s_t)\pi_t + \gamma_2(s_t)x_t \right] + \epsilon_{R,t}, \]

where \( x_t \) is the output gap \( t \), \( \pi_t \) is the inflation rate and \( R_t \) is the nominal interest rate with both \( \pi_t \) and \( R_t \) are measured in terms of deviations from the regime-dependent state. The shocks to the consumer and firm’s sectors evolve as

\[
\begin{bmatrix}
    z_{D,t} \\
    z_{S,t}
\end{bmatrix} = \begin{bmatrix}
    \rho_D(s_t) & 0 \\
    0 & \rho_S(s_t)
\end{bmatrix} \begin{bmatrix}
    z_{D,t-1} \\
    z_{S,t-1}
\end{bmatrix} + \begin{bmatrix}
    \epsilon_{D,t} \\
    \epsilon_{S,t}
\end{bmatrix},
\]
Lubik and Schorfheide estimated the model assuming that the agents behaved as if each regime were to last indefinitely. The parameter estimates are given below:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Structural Equations</th>
<th>Shock Processes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\tau$ $\kappa$ $\beta$ $\gamma_1$ $\gamma_2$</td>
<td>$\rho_D$ $\rho_S$ $\rho_R$ $\sigma_D$ $\sigma_S$ $\sigma_R$</td>
</tr>
<tr>
<td>First regime</td>
<td>0.69 0.77 0.997 0.77 0.17</td>
<td>0.68 0.82 0.60 0.27 0.87 0.23</td>
</tr>
<tr>
<td>Second regime</td>
<td>0.54 0.58 0.993 2.19 0.30</td>
<td>0.83 0.85 0.84 0.18 0.37 0.18</td>
</tr>
</tbody>
</table>
Example (continued)

- Considering each regime in isolation, the first regime would be indeterminate while the second regime is determinate.
- Given the above calibration, one could compute the MSV solution, which would be different from the solutions obtained by considering each regime in isolation.
- Given any parameterization, one can find MSV solutions of the model (assuming that any exist), and could use the solution to approximate the likelihood.
- These techniques could thus be used to estimate the model taking into account that agents know that the regimes could switch.
Conclusions

• We extend the notion of the MSV solution (MacCallum (1983)) to Markov-switching rational expectations models.

• We characterize both the general and MSV solutions of linear rational expectations model with Markov-switching parameters.

• We derive an easy to use and efficient algorithm for obtaining MSV solutions.