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Volatility forecasting: the jumps do matter[*][†]

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Abstract

This study reconsiders the role of jumps for volatility forecasting by showing that jumps have a positive and mostly significant impact on future volatility. This result becomes apparent once volatility is correctly separated into its continuous and discontinuous component. To this purpose, we introduce the concept of threshold multipower variation (TMPV), which is based on the joint use of bipower variation and threshold estimation. With respect to alternative methods, our TMPV estimator provides less biased and robust estimates of the continuous quadratic variation and jumps. This technique also provides a new test for jump detection which has substantially more power than traditional tests. We use this separation to forecast volatility by employing an heterogeneous autoregressive (HAR) model which is suitable to parsimoniously model long memory in realized volatility time series. Empirical analysis shows that the proposed techniques improve significantly the accuracy of volatility forecasts for the S&P500 index, single stocks and US bond yields, especially in periods following the occurrence of a jump.

Keywords: volatility forecasting, jumps, bipower variation, threshold estimation, stock, bond.

JEL Classification codes: G1,C1,C22,C53

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1 Introduction

The importance of jumps in financial economics is widely recognized. A partial list of recent studies on this topic includes test specification of Aït-Sahalia (2004), Jiang and Oomen (2006), Barndorff-Nielsen and Shephard (2006), Lee and Mykland (2007) and Aït-Sahalia and Jacod (2008), as well as the empirical studies of Bollerslev, Law and Tauchen (2007) and Maheu and McCurdy (2004); Bollerslev et al. (2007); Andersen et al. (2006); nonparametric estimation in the presence of jumps, as in Bandi and Nguyen (2003); Johannes (2004); Mancini and Renò (2006); option pricing as in Duffie et al. (2002); Eraker et al. (2003); Eraker (2004); applications to bond and stock market, as in Das and Uppal (2004); Wright and Zhou (2007). Interesting references for a review are Cont and Tankov (2004) and Barndorff-Nielsen and Shephard (2007). However, while jumps have been shown to be relevant in economic and financial applications, they still have no direct role for volatility forecasting.

In this study we start from an apparent puzzle contained in the study of Andersen et al. (2007) (henceforth ABD) and Forsberg and Ghysels (2007); Giot and Laurent (2007); Busch et al. (2006). In these works, jumps have been found to possess a negative or null impact in determining future volatility. We find this result puzzling in at least two respects. First, visual inspection of realized volatility time series reveals that bursts in volatility are usually initiated by a large and unexpected movement of asset prices; this suggests that jumps should have a forecasting power for volatility. Second, it is well known that volatility is associated with dispersion of beliefs and heterogeneous information, see e.g. Shalen (1993); Wang (1994) and Buraschi et al. (2007). If the occurrence of a jump increases the uncertainty on fundamental values, it is likely to have a positive impact on future volatility.

This paper provides four contributions to the literature. The first contribution is to show that this puzzle is due to the fact that preceding studies in the literature use bipower variation to estimate the continuous integrated volatility and, by difference, the jump contribution to total quadratic variation. While bipower variation is a consistent estimator of integrated volatility as the time interval between observation vanishes, our simulation studies show that in finite samples it is largely upper biased in presence of jumps, and this implies a large underestimation of the jump component. Unfortunately, this problem cannot be accommodated by simply shrinking the observation interval, since market microstructure effects would jeopardize the estimation in an unpredictable way. [1]

An alternative estimator of integrated volatility has been introduced by Mancini (2007) and studied by

[1] Attempts to study and correct bipower variation under microstructure noise can be found in Podolskij and Vetter (2006), and Christensen et al. (2008). ABD and Huang and Tauchen (2005) propose a staggered version of bipower variation. Fan and Wang (2007) study jumps and microstructure noise with wavelet methods. The impact of microstructure noise on threshold estimation of Mancini (2007) is instead unknown. The results in this paper can potentially be extended to account for microstructure noise. Directly incorporating microstructure noise can improve volatility forecasting, see e.g. Aït-Sahalia and Mancini (2008).

Aït-Sahalia and Jacod (2007) in a broader context. This estimator is potentially less biased than bipower variation, but it requires the auxiliary estimation of a threshold function; thus, when it is used in its original form, it provides estimates of continuous volatility which are sensitive to the specification of the threshold.

The second contribution of this paper is thus to introduce an alternative estimator of integrated powers of volatility in presence of jumps. We introduce (Section 2) the concept of *threshold multipower variation*, which can be viewed as combination of the above mentioned techniques. Using realistic simulation of asset prices (Section 3), we show that threshold bipower variation is nearly unbiased on continuous trajectories and, importantly, also in the presence of jumps. Moreover, it is robust to the choice of the threshold function, in the sense that the impact of the threshold on estimation is marginal. Thus, it is an ideal candidate to estimate dynamic models of volatility in which we use separately the continuous and discontinuous volatility as explanatory variables.

Our third contribution is the introduction of a novel test for jump detection in time series. Our test is basically a correction of the z statistics of Barndorff-Nielsen and Shephard (2006) based on threshold multipower variation. This correction removes the bias in estimating the integral of the second and fourth power of continuous volatility in presence of jumps. We show that our C-Tz test is sized as the traditional z-test under the null. In presence of jumps instead, the corrected test has significantly more power than the z test, especially when jumps are consecutive, a situation which is quite frequent in high-frequency data. We show this on simulated data (Section 3) and on time series which are well known to display very large jumps (Section 4), namely electricity prices and interest rate data, for which the z-test has a disappointingly low power.

We use the separation of the quadratic variation in its continuous and discontinuous component in a suitable extension of the forecasting model of Corsi (2004), which, in spite of its simplicity, is able to reproduce the main features of volatility dynamics, including long memory (Section 5).

Empirical results (Section 6) confirm our theoretical results and constitute our fourth contribution. We work on stock index futures, single stocks and treasury bond data. On equity data, we find that jumps have a positive and highly significant impact on future volatility, a result which cannot be observed when bipower variation is employed because of its inherent bias. Uniformly on our data sets, our forecasting models provide higher R^2, especially in days following a jump, and lower root mean square error, with respect to the existing models in the literature.

Concluding remarks are in Section 7.

2 Disentangling diffusion from jumps

2.1 Introductory concepts

We work in a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{F}, \mathcal{P})$, satisfying the usual conditions (Protter, 1990). We assume that an economic variable X_t, for example the logarithmic price of a stock or an interest rate, satisfies the following assumption:

Assumption 2.1 $(X_t)_{t\in[0,T]}$ *is a real-valued process such that* $X_0 \in \mathbb{R}$ *and*

$$dX_t = \mu_t dt + \sigma_t dW_t + c_t dN_t \tag{2.1}$$

where μ_t is predictable, σ_t is cádlág and N_t is a Poisson process whose intensity is an adapted stochastic process λ_t, the times of the corresponding jumps are $(\tau_j)_{j=1,\ldots,N_T}$ and c_j are i.i.d. adapted random variables measuring the size of the jump at time τ_j.

Typically, in financial econometrics a time window T is fixed, e.g. one day, and we define the quantities of interest on a time span of length T. Quadratic variation of such a process over a time window is defined as:

$$[X]_t^{t+T} := X_{t+T}^2 - X_t^2 - 2\int_t^{t+T} X_{s-} dX_s, \tag{2.2}$$

where t indexes the interval, typically a day, and it can be decomposed into its continuous and discontinuous component, as:

$$[X]_t^{t+T} = [X^c]_t^{t+T} + [X^d]_t^{t+T} \tag{2.3}$$

where $[X^c]_t^{t+T} = \int_t^{t+T} \sigma_s^2 ds$ and $[X^d]_t^{t+T} = \sum_{j=N_t}^{N_{t+T}} c_{\tau_j}^2$, where c_{τ_j} is the size of the j-th jump at time τ_j. To estimate these quantities, we divide the time interval $[t, t+T]$ into n subintervals of length δ, thus $\delta = T/n$. On this grid, we define the evenly sampled returns as:

$$\Delta_{j,t} X = X_{j\delta+t} - X_{(j-1)\delta+t}, \qquad j = 1, \ldots, n \tag{2.4}$$

For simplicity, in what follows we omit the subscript t and we simply write $\Delta_j X$. An estimator of $[X]_t^{t+T}$ is given by realized volatility, defined as:

$$\mathsf{RV}_\delta(X)_t = \sum_{j=1}^{n} (\Delta_j X)^2 \underset{\delta\to 0}{\longrightarrow} [X]_t^{t+T} \tag{2.5}$$

where the above convergence is in probability.

To disentangle the continuous quadratic variation from the discontinuous one, multipower variation has been introduced by Barndorff-Nielsen and Shephard (2006) and it is defined as:

$$\mathsf{MPV}_\delta(X)_t^{[\gamma_1,\ldots,\gamma_M]} = \delta^{1-\frac{1}{2}(\gamma_1+\ldots+\gamma_M)} \sum_{j=M}^{[T/\delta]} \prod_{k=1}^{M} |\Delta_{j-k+1} X|^{\gamma_k} \tag{2.6}$$

It is a natural extension of the concept of bipower variation, studied in Barndorff-Nielsen and Shephard (2004); Barndorff-Nielsen et al. (2006). Asymptotic properties of multipower variation have been studied by Barndorff-Nielsen et al. (2006) in absence of jumps, and by Barndorff-Nielsen, Shephard and Winkel (2006) and Woerner (2006) in presence of jumps. With very mild assumptions, and in some cases for infinite activity Levy jump processes, Barndorff-Nielsen, Shephard and Winkel (2006) show that:

$$p - \lim_{\delta \to 0} \mathsf{MPV}_\delta(X)_t^{[\gamma_1,\ldots,\gamma_M]} = \left(\prod_{k=1}^{M} \mu_{\gamma_k} \right) \int_t^{t+T} \sigma_s^{\gamma_1+\ldots+\gamma_M} ds \tag{2.7}$$

where $\mu_\gamma = E(|u|^\gamma) = 2^{\gamma/2} \frac{\Gamma(\frac{\gamma+1}{2})}{\Gamma(1/2)}$, with $u \sim \mathcal{N}(0,1)$, and the above convergence is in probability. For practical applications, multipower variation is used for the estimation of $\int_t^{t+T} \sigma_s^2 ds$ and $\int_t^{t+T} \sigma_s^4 ds$. For this motivation, we define some relevant special cases of multipower variation, according to suitable choices of the vector $\gamma_1, \ldots, \gamma_M$. The most important example is bipower variation:

$$\mathsf{BPV}_\delta(X)_t = \mu_1^{-2}\, \mathsf{MPV}_\delta(X)_t^{[1,1]} = \mu_1^{-2} \sum_{j=2}^{[T/\delta]} |\Delta_{j-1}X| \cdot |\Delta_j X| \underset{\delta \to 0}{\longrightarrow} \int_t^{t+T} \sigma_s^2 ds \tag{2.8}$$

with $\mu_1 \simeq 0.7979$. For estimators of $\int_t^{t+T} \sigma_s^4 ds$, see Appendix A.

Based on a threshold function $\Theta(\delta)$, Mancini (2007) provides alternative estimators to the multipower variation ones, for squared and fourth power volatility. These estimators are defined as follows:

$$\mathsf{TRV}_\delta(X)_t = \sum_{j=1}^{[T/\delta]} |\Delta_j X|^2 I_{\{|\Delta_j X|^2 \le \Theta(\delta)\}} \underset{\delta \to 0}{\longrightarrow} \int_t^{t+T} \sigma_s^2 ds \tag{2.9}$$

$$\mathsf{TQV}_\delta(X)_t = \frac{1}{3\delta} \sum_{j=1}^{[T/\delta]} |\Delta_j X|^4 I_{\{|\Delta_j X|^2 \le \Theta(\delta)\}} \underset{\delta \to 0}{\longrightarrow} \int_t^{t+T} \sigma_s^4 ds \tag{2.10}$$

where the above convergence is in probability, and where the threshold functions has to satisfy

$$\lim_{\delta \to 0} \Theta(\delta) = 0, \qquad \lim_{\delta \to 0} \frac{\delta \log \frac{1}{\delta}}{\Theta(\delta)} = 0, \tag{2.11}$$

that is they have to vanish slower than the modulus of continuity of the Brownian motion. Mancini (2007) also establishes a central limit theorem for TRV.

2.2 Threshold multipower variation

We now introduce the following extension of multipower variation. In what follows, we use a strictly positive threshold function $\vartheta_s : [t, t+T] \to \mathbb{R}^+$, which, contrary to the threshold function used in (2.9), does not depend on δ. For brevity, we denote by $\vartheta_j = \vartheta_{j\delta}$. We define the threshold multipower variation estimator as follows:

Definition 2.2 *We define the (realized) threshold multipower variation as:*

$$\mathsf{TMPV}_\delta(X)_t^{[\gamma_1,\ldots,\gamma_M]} = \delta^{1-\frac{1}{2}(\gamma_1+\ldots+\gamma_M)} \sum_{j=M}^{[T/\delta]} \prod_{k=1}^{M} |\Delta_{j-k+1}X|^{\gamma_k} I_{\{|\Delta_{j-k+1}X|^2 \le \vartheta_{j-k+1}\}} \tag{2.12}$$

The intuition behind the concept of threshold multipower variation is the following. Suppose $\Delta_j X$ contains a jump. In the case of bipower variation, it will multiply two adjacent returns, $\Delta_{j-1}X$ and $\Delta_{j+1}X$. Asymptotically, both these returns will vanish and bipower variation will converge to integrated continuous volatility. But for finite δ these returns will not vanish, causing a positive bias which will be larger as $\Delta_j X$ increases. This consideration suggests that the bias of multipower variation will be extremely large in case of consecutive jumps, as will be shown in Section 4. For estimators (2.12) instead, if $\Delta_j X$ contains a jump larger than ϑ_j, the corresponding indicator function vanishes, thus correcting for the bias. This intuition is supported by the analysis in the subsequent sections.

Formally, we can state the following Theorem regarding asymptotic behavior of threshold multipower variation:

Theorem 2.3 *Assume 2.1 holds, and that θ_t is a real positive mapping defined on $[0, T]$. Then, as $\delta \to 0$,*

1. *if $\max(\gamma_1, \ldots, \gamma_M) < 2$ we have:*

$$\mathsf{TMPV}_\delta(X)_t^{[\gamma_1,\ldots,\gamma_M]} \longrightarrow \left(\prod_{k=1}^{M} \mu_{\gamma_k}\right) \int_t^{t+T} \sigma_s^{\gamma_1+\ldots+\gamma_M} ds \tag{2.13}$$

the above convergence is in probability.

2. *if $\max(\gamma_1, \ldots, \gamma_M) < 1$ we have:*

$$\delta^{-\frac{1}{2}} \left(\mathsf{TMPV}_\delta(X)_t - \int_t^{t+T} \sigma_s^{\gamma_1+\ldots+\gamma_M} ds\right) \longrightarrow c_\gamma \int_t^{t+T} \sigma_s^{\gamma_1+\ldots+\gamma_M} dW'_s \tag{2.14}$$

where W' is a Brownian motion independent on W, the above convergence holds stably in law and:

$$c_\gamma^2 = \prod_{k=1}^{M} \mu_{2\gamma_k} - 2(M-1) \prod_{k=1}^{M} \mu_{\gamma_k}^2 + 2 \sum_{j=1}^{M-1} \left(\prod_{k=1}^{j} \mu_{\gamma_k} \prod_{k=M-j+1}^{M} \mu_{\gamma_k} \prod_{k=1}^{M-j} \mu_{\gamma_k+\gamma_{k+j}} \right) \tag{2.15}$$

Proof. Under Assumption 2.1, we write:

$$X = Y + Z$$

where $Y_t = \int_t^{t+T} \mu_s ds + \int_t^{t+T} \sigma_s dW_s$. If $Z = 0$, the Theorem has been proved by Barndorff-Nielsen et al. (2006). Since Z is a finite activity jump process and Y is continuous, for every trajectory there exists δ' such that the number of returns such that $|\Delta X_j|^\gamma > \vartheta_j$ is finite. For $\delta < \delta'$, the difference $\mathsf{TMPV}_\delta(X)_t^{[\gamma_1,\ldots,\gamma_M]} - \mathsf{TMPV}_\delta(Y)_t^{[\gamma_1,\ldots,\gamma_M]}$ is zero for those terms who do not contain jumps; in the remainder every term contains at most one jump, which is summed up over M finite terms, with only one term containing a jump. For the others, uniformly on j we have:

$$\delta^{-\frac{1}{2}} |\Delta_j X| = O_p(|\log \delta|^{\frac{1}{2}}).$$

Thus the difference is $O_p\left(\left(\delta\log\frac{1}{\delta}\right)^{\frac{1}{2}(\sum_{i=1}^M\gamma_i-\max(\gamma_1,\ldots,\gamma_M))}\right)$. Thus

$$\delta^{1-\sum_{i=1}^M\gamma_i}\left(\mathsf{TMPV}_\delta(X)_t^{[\gamma_1,\ldots,\gamma_M]}-\mathsf{TMPV}_\delta(Y)_t^{[\gamma_1,\ldots,\gamma_M]}\right)=$$

$$=O_p\left(\delta^{1-\frac{1}{2}\max(\gamma_1,\ldots,\gamma_M)}\left(\log\frac{1}{\delta}\right)^{\frac{1}{2}(\sum_{i=1}^M\gamma_i-\max(\gamma_1,\ldots,\gamma_M))}\right)$$

The latter term is at least $O_p(1)$ if $\max(\gamma_1,\ldots,\gamma_M)<2$, which implies that $\mathsf{TMPV}_\delta(X)_t^{[\gamma_1,\ldots,\gamma_M]}$ has the same limit in probability of $\mathsf{TMPV}_\delta(Y)_t^{[\gamma_1,\ldots,\gamma_M]}$; and it has the same limit in law if $\max(\gamma_1,\ldots,\gamma_M)<1$, since in this case the difference is $O_p(\delta^{\frac{1}{2}})$. □

In Theorem 2.3 we can also allow for infinite activity jumps for suitable conditions on the coefficients $\gamma_1,\ldots,\gamma_M$, see e.g. Jacod (2008). Then, not surprisingly, threshold multipower variation has the same distribution of multipower variation and threshold power variation when $\delta\to 0$. However, our interest is in estimating $\int\sigma_s^2 ds$ when δ is finite and large enough to avoid microstructure effects (typically, $\delta=5$ minutes), and in this case we will show that the three estimators are different.

As a special case for estimating integrated variance, we introduce threshold bipower variation as follows:

$$\mathsf{TBPV}_\delta(X)_t=\mu_1^{-2}\,\mathsf{TMPV}_\delta(X)_t^{[1,1]}=\mu_1^{-2}\sum_{j=2}^{[T/\delta]}|\Delta_{j-1}X|\cdot|\Delta_jX|I_{\{|\Delta_{j-1}X|^2\le\vartheta_{j-1}\}}I_{\{|\Delta_jX|^2\le\vartheta_j\}}\tag{2.16}$$

For estimators of the integrated fourth-power of volatility, see Appendix A.

Now, we describe our technique to select a suitable threshold, and show how to implement threshold multipower variation when δ is finite.

2.3 Tuning the threshold function

In the forthcoming sections, we will show that the choice of the threshold function is almost immaterial for our purposes. Thus, for simplicity we use a threshold function which is defined as a multiple of the local variance, to approximate which we use a local linear filter of length $2L+1$ (Fan and Yao, 2003) adapted for the presence of jumps by iterating in Z:

$$\hat{V}_t^Z=\frac{\displaystyle\sum_{i=-L,\ i\neq-1,0,1}^{L}K\left(\frac{i}{L}\right)(\Delta_{t+i}X)^2\,I_{\{(\Delta_{t+i}X)^2\le c_V^2\cdot\hat{V}_{t+i}^{Z-1}\}}}{\displaystyle\sum_{i=-L,\ i\neq-1,0,1}^{L}K\left(\frac{i}{L}\right)I_{\{(\Delta_{t+i}X)^2\le c_V^2\cdot\hat{V}_{t+i}^{Z-1}\}}},\qquad Z=1,2,\ldots\tag{2.17}$$

with the starting value set to $\hat{V}^0=+\infty$, which corresponds to using all observations in the first step, and $c_V=3$. Asymptotic properties for a class of related estimators of local variance in absence of jumps can be found in Kristensen (2007).

At each iteration, jumps are detected by the condition $(\Delta_t X)^2 > c_V^2 \cdot \hat{V}_t^{Z-1}$ and removed from the time series by means of the indicator function[2]; each estimate of the variance is multiplied by c_V^2 to get the threshold for the following step. For estimation of the local variance at time t, we do not use the adjacent observations ($i \neq -1, 0, 1$). The iterations stop when the removed jumps are the same. On high frequency data, this always happens with $Z = 2, 3$ iterations.

The bandwidth parameter L determines the number of adjacent returns included in the estimation of the local variance around point t. In our application, its choice results not to be crucial. We set $L = 25$. As usual, the choice of the kernel function $K(\cdot)$ is quite uninfluential in this kind of applications (Silverman, 1986; Wand and Jones, 1995). We use a Gaussian kernel:

$$K(y) = \left(1/\sqrt{2\,\pi}\right) \exp\left(-y^2/2\right). \tag{2.18}$$

Then we set the threshold function proportional to the local variance

$$\vartheta_t = c_\vartheta^2 \cdot \hat{V}_t^Z \tag{2.19}$$

A typical value of c_ϑ is $c_\vartheta = 3$. However, the dimensionless parameter c_ϑ can be used to scale the threshold with respect to local variance, and by varying it we can test the robustness of proposed estimators with respect to the choice of the threshold.

2.4 A corrected test for jump detection

While threshold multipower variation has the same asymptotic law of multipower variation, we expect it to provide better estimates in small samples. However, for finite δ , when $|\Delta X|^2 > \vartheta$ the indicator function in (2.12) zeroes its relative addend. This can be an issue when testing for the presence of jumps, for example with the z statistics introduced by Barndorff-Nielsen and Shephard (2006), since under the null of no jumps there are still variations larger than the threshold. These variations are also the larger ones, thus TMPV is expected to be negatively biased under the null. However, this issue can be solved by correcting the estimator according to the following rule: when $|\Delta X|^2 > \vartheta$ we replace $|\Delta X|^\gamma$ with its expected value under the null (instead of 0). Using the fact that, when X is $\mathcal{N}(0, \sigma^2)$, we have:

$$\mathbb{E}\left[|X|^\gamma \,\middle|\, X^2 > \vartheta\right] = \frac{\Gamma\left(\frac{\gamma+1}{2}, \frac{c_\vartheta^2}{2}\right)}{2N(-\frac{\sqrt{\vartheta}}{\sigma})\sqrt{\pi}} \left(2\sigma^2\right)^{\frac{1}{2}\gamma}, \tag{2.20}$$

where $N(x)$ is the standard normal cumulative function and $\Gamma(\alpha, x)$ is the upper incomplete gamma function.[3] Now, we can exploit the estimate of the continuous local volatility given by the threshold itself, which is

[2] Alternatively, one may use the approach proposed by Lee and Mykland (2007) to detect and remove individual jumps and subsequently estimate the local variance with the remaining observations.

[3] Precisely,

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} ds$$

given by (2.19), replacing σ^2 by ϑ/c_ϑ^2 in (2.20). Then, we define the corrected realized threshold multipower estimator as:

$$\mathsf{C\text{-}TMPV}_\delta(X)_t^{[\gamma_1,\ldots,\gamma_M]} = \delta^{1-\frac{1}{2}(\gamma_1+\ldots+\gamma_M)} \sum_{j=M}^{[T/\delta]} \prod_{k=1}^{M} Z_{\gamma_k}(\Delta_{j-k+1}X, \vartheta_{j-k+1}) \tag{2.21}$$

where the function $Z_\gamma(x,y)$ is defined as:

$$Z_\gamma(x,y) = \begin{cases} |x|^\gamma & if\ x^2 \le y \\ \dfrac{1}{2N(-c_\vartheta)\sqrt{\pi}} \left(\dfrac{2}{c_\vartheta^2} y\right)^{\frac{\gamma}{2}} \Gamma\left(\dfrac{\gamma+1}{2}, \dfrac{c_\vartheta^2}{2}\right) & if\ x^2 > y \end{cases} \tag{2.22}$$

Relevant cases which will be examined in what follows are $\gamma = 1, 2, 4/3$. In these special cases we have, with $c_\vartheta = 3$ and $x^2 > y$, $Z_1(x,y) \simeq 1.094 \cdot y^{\frac{1}{2}}$, $Z_{4/3}(x,y) \simeq 1.129 \cdot y^{\frac{2}{3}}$, and $Z_2(x,y) \simeq 1.207 \cdot y$ respectively.

For example, the corrected version of (2.16) is the corrected threshold bipower variation defined as:

$$\mathsf{C\text{-}TBPV}_\delta(X)_t = \mu_1^{-2}\, \mathsf{C\text{-}TMPV}_\delta(X)_t^{[1,1]} = \mu_1^{-2} \sum_{j=2}^{[T/\delta]} Z_1(\Delta X_j, \vartheta_j) Z_1(\Delta X_{j-1}, \vartheta_{j-1}) \tag{2.23}$$

As will be shown in the next section, the correction is essential for building test statistics; indeed, the correction provides unbiased estimates under the null. The test statistics which is used in our empirical analysis is based on this correction and it is defined by:

$$\mathsf{C\text{-}Tz} = \delta^{-\frac{1}{2}} \frac{\left(\mathsf{RV}_\delta\,(X)_T - \mathsf{C\text{-}TBPV}_\delta\,(X)_T\right) \cdot \mathsf{RV}_\delta\,(X)_T^{-1}}{\sqrt{\left(\frac{\pi^2}{4} + \pi - 5\right) \max\left\{1, \frac{\mathsf{C\text{-}TTriPV}_\delta(X)_T}{\left(\mathsf{C\text{-}TBPV}_\delta(X)_T\right)^2}\right\}}}, \tag{2.24}$$

In light of Theorem 2.3, since for small δ the correction affects only a finite number of terms, we have that $\mathsf{C\text{-}Tz} \to \mathcal{N}(0,1)$ stably in law as $\delta \to 0$.

A correction similar to (2.21) can also be introduced for the threshold estimators of Mancini (2007). We define:

$$\mathsf{C\text{-}TPV}_\delta(X)_t^{[\gamma]} = \delta^{1-\frac{1}{2}\gamma} \sum_{j=1}^{[T/\delta]} Z_\gamma(\Delta_j X, \vartheta_j). \tag{2.25}$$

While the corrected version (2.21) of threshold bipower variation is expected to be unbiased in absence of jumps, it will introduce a positive bias if jumps are present in the trajectory of X. Indeed, suppose that there is a jump ΔJ in a given interval, such that $\Delta X = \Delta X^c + \Delta J$. When detecting the jump, we approximate $(\Delta X)^\gamma$ with $\mathbb{E}\left[|X^c|^\gamma \,\middle|\, (X^c)^2 > \vartheta\right]$, which is much larger than $\mathbb{E}\left[|X^c|^\gamma\right]$, which is our estimation target. Thus, the correction should not be implemented when the test statistics detects a jump in the trajectory.

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and

$$\Gamma(\alpha, x) = \int_x^{+\infty} s^{\alpha-1} e^{-s} ds.$$

When $\alpha = 1$, $\Gamma(1,x) = e^{-x}$. For large γ, we can exploit the integration by parts formula $\Gamma(\alpha+1,x) = \alpha\Gamma(\alpha,x) + x^\alpha e^{-x}$.

Parameter	Value
μ	0.0304
α	-0.0120
β	0.0145
η	0.1153
ρ	-0.6127
σ_J	0.0151

Table 1: Parameters of model (3.1) expressed in percentage form and on daily basis. The intensity λ is not reported, since it is not used in our simulations.

3 Simulation study

To assess the small sample properties of the concurrent estimators we use Monte Carlo simulations of realistic stochastic processes which have been extensively used to model stock index prices. The purpose of this section is to show that bipower variation is a biased estimator of integrated volatility in the presence of jumps, while threshold-based estimator, both power and multipower, are much less sensitive to jumps and accordingly less biased. Moreover, we show that while threshold power variation is particularly sensitive to the choice of the threshold, threshold bipower variation is instead largely robust to this choice. This latter feature is particularly important, since it suggests that results obtained in empirical applications using threshold multipower variation are not too sensitive to the threshold employed.

The model we simulate is a one-factor jump-diffusion model with stochastic volatility, described by the couple of stochastic differential equations:

$$\begin{array}{lcl} dX_t & = & \mu\, dt + \sqrt{v_t} dW_{x,t} + c_t dN_t, \\ d\log v_t & = & (\alpha - \beta \log v_t)\, dt + \eta dW_{v,t}, \end{array} \tag{3.1}$$

where W_x and W_v are standard Brownian motions with $\mathrm{corr}\,(dW_x, dW_v) = \rho$, v_t is a stochastic volatility factor, $c_t dN_t$ is a compound Poisson process with constant intensity λ and random jump size which is Normally distributed with zero mean and standard deviation σ_J. We use the model parameters estimated by Andersen et al. (2002) on $S\&P500$ prices and reported in Table 1. Similar estimates have been obtained by Bates (2000); Pan (2002); Chernov et al. (2003).

The numerical integration of model (3.1) is performed with the Euler scheme, using a discretization step of $\Delta = 1$ second. Each day, we simulate $7 \cdot 60 \cdot 60$ steps corresponding to seven hours. We then use $\delta = 5$ minutes, that is 84 returns per day.

The Monte Carlo experiments are devised to compare the efficiency of the proposed estimators in estimating

Quantity	Estimator	Relative bias (%)			
		no jumps	one jump	two jumps	two consecutive jumps
	BPV	-0.90 (0.55)	47.59 (1.78)	110.73 (4.78)	659.35 (25.86)
	TRV	-5.09 (0.53)	-7.61 (0.51)	-7.81 (0.52)	-7.77 (0.51)
$\int \sigma_s^2 ds$	C-TRV	-0.22 (0.48)	7.64 (0.54)	17.89 (0.60)	17.92(0.60)
	TBPV	-5.16 (0.59)	-8.74 (0.58)	-10.00 (0.56)	-9.31 (0.56)
	C-TBPV	-1.09 (0.55)	3.23 (0.59)	9.91 (0.63)	19.03 (0.68)
	QPV	-2.72 (1.29)	88.97 (5.18)	284.79 (23.83)	1939.80 (118.49)
	TQV	-15.47 (0.99)	-18.50 (0.92)	-17.65 (0.96)	-17.73 (0.94)
	C-TQV	-1.22 (1.06)	34.08 (1.46)	74.60 (1.95)	73.95 (1.97)
$\int \sigma_s^4 ds$	TQPV	-9.29 (1.29)	-14.69 (1.24)	-18.15 (1.20)	-16.46 (1.18)
	C-TQPV	-3.00 (1.29)	7.02 (1.51)	21.51 (1.72)	49.81 (2.53)
	TriPV	-1.79 (1.26)	208.83 (12.06)	754.97 (89.27)	10097.0 (707.38)
	TTriPV	-9.50 (1.24)	-14.05 (1.17)	-16.07 (1.13)	-14.91 (1.12)
	C-TTriPV	-2.33 (1.26)	9.63 (1.46)	26.19 (1.71)	71.32 (2.81)

Table 2: Reports the mean error in percentage form in estimating $\int \sigma_s^\gamma ds$, with $\gamma = 2, 4$, using the corresponding estimator, in the case of no jumps, one jump, two jumps and two consecutive jumps when simulating model (3.1). In parenthesis, the standard error of the mean is reported.

the integral of σ^2 and σ^4. Since we are keenly interested in studying the performance of competing estimators in presence of jumps, we generate different samples (with $1,000$ "daily" replications each) in the following way. In the first sample, we do not generate jumps at all. In the second sample, we generate a single jump for each day. In the third sample, we generate exactly two jumps per day. In the fourth sample, we generate two jumps per day and we force them to be consecutive (i.e., the second jump is forced to occur 300 seconds after the first). This allows us to compute the expected value conditioned to the presence of zero, one, two jumps, consecutive or not. For every simulated daily trajectory, we compute the estimates of BPV and their fourth-power counterparts QPV,TriPV as well as threshold estimators TRV, TQV, TTriPV and threshold multipower estimators TBPV, TQPV,TTriPV. All these estimators are precisely defined in Appendix A. We compute daily percentage estimation error and compute averages and standard deviations across the sample. All results are reported in Table 2.

Results are compelling. Bipower variation (as realized volatility) does a good job in estimating integrated squared volatility in case of no jumps. However, it is significantly biased if there is a jump in the trajectory (+47.59%) and largely biased (+110.73%) if there are two jumps in the trajectory. If the two jumps are consecutive, the bias is huge (+659.35%). The bias of multipower variation in estimating integrated quarticity is even larger.

Threshold-based estimators, instead, are much more robust to the presence of jumps. The bias of threshold power variation of Mancini (2007) in estimating integrated squared volatility is around -5% in absence of jumps and around -8% in presence of one and two jumps, consecutive or not. The same happens when estimating quarticity, the bias being around -15%. The presence of a negative bias is due to the fact that, by their proper definition, threshold estimators remove completely observations larger than the threshold. When we correct for this as indicated in Section 2.4, the bias turns out to be positive since, when an observation is above the threshold, we replace its power with its expected value under the assumption that the observation was actually above threshold; which is true under the null of no jumps, but needs not to be true in presence of an actual jump, thus inducing a positive bias.

The estimators based on threshold multipower variation, introduced in this study, yield equally good results. Threshold bipower variation has a bias of -5.16% in the case of no jumps, of -8.74% with a single jump, of -10.0% in the case of two jumps, and of -9.31% in the case of two consecutive jumps. When estimating quarticity, these biases range between -9% and -15% according to the number of jumps and the estimator used. Again, the corrected versions largely correct the bias under the null of no jumps, but turn the negative bias in a positive one in the case of jumps. However, from our simulated experiment we can conclude that threshold-based estimators perform much better than multipower variation in presence of jumps.

Threshold estimators deliver more accurate estimates in the presence of jumps at the cost of introducing this extra-parameter (the results presented above are based on the threshold (2.19) with $c_\vartheta = 3$). Between the two competing threshold estimators (power and multipower), our simulation experiments highlight a substantial advantage in using threshold multipower variation instead of threshold power variation. This conclusion stems from considering the robustness of jump detection and volatility measuring with respect to the parameter c_ϑ. We use simulated experiments to check the robustness of threshold estimators with respect to this parameter, in the case of a single jump and in the case of two jumps. Results are shown in Figure 1.

Bipower variation does not depend on the value of the threshold but it is largely biased, especially with two jumps in the daily trajectory. Threshold estimators are less biased, however we can see that threshold bipower variation is less sensitive to the choice of the threshold than threshold power variation. This is basically due to the fact that, even if both TBPV and TRV converge to $[X^c]$ as $\delta \to 0$, for fixed δ we have $\mathsf{TBPV}_\delta \underset{c_\vartheta \to \infty}{\longrightarrow} \mathsf{BPV}_\delta$ while $\mathsf{TRV}_\delta \underset{c_\vartheta \to \infty}{\longrightarrow} \mathsf{RV}_\delta$.

We also use Monte Carlo experiments to evaluate the efficiency in detecting jumps with the z statistics (A.7), constructed with multipower variation methods, the Tz statistics (A.12), constructed with threshold multipower variation and the $\mathsf{C\text{-}Tz}$ statistics (2.24), corrected as explained in Section 2.4, as well as with the corresponding logarithmic test statistics defined in Appendix A.

Results with different confidence levels are reported in Table 3 in the case of no jumps, a single jump, and

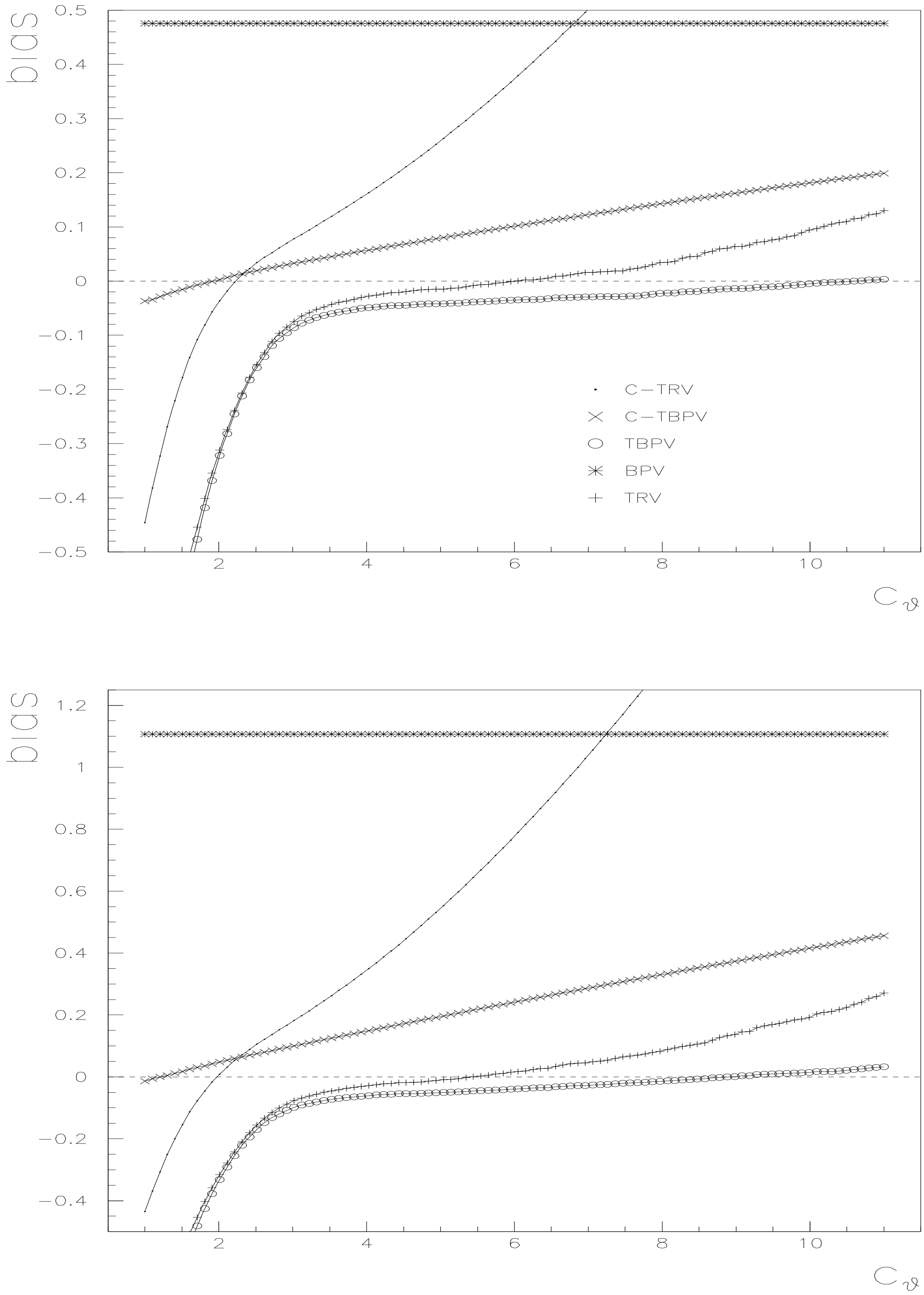

Figure 1: Bias of the different estimators of $\int \sigma_s^2 ds$ in the presence of a single jump (top) and two jumps (bottom), as a function of the threshold parameter c_ϑ.

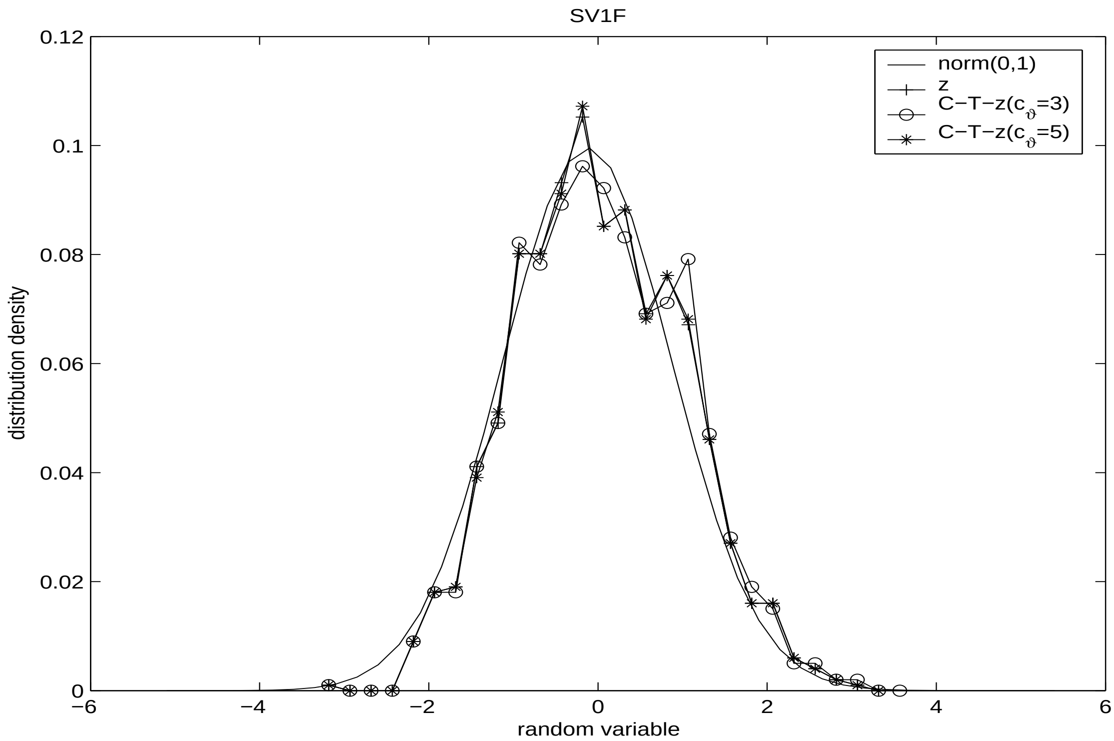

Figure 2: Distribution of the jump statistics z and C-Tz (with $c_\vartheta = 3, 5$) on 1,000 replications of model (3.1) with $cdN = 0$.

two consecutive jumps, while Figure 2 shows the distribution of selected test statistics under the null.

The efficiency of detecting jumps with the Tz statistics is larger than that obtained with z statistics; however also the noise is larger and well above the values expected by the significance level required. This comes from the fact that, under the null of no jumps that we are testing, it is better to use C-Tz statistics. Indeed, Table 3 shows that the C-Tz (with $c_\vartheta = 3$) statistics yield a noise which is almost identical to that of the z statistics still preserving a detection efficiency larger than that of the z statistics. Figure 2 shows the distribution of the statistics under the null of no jumps, and we can see that the behavior on that sample of the z and the C-Tz statistics, with both $c_\vartheta = 3$ and $c_\vartheta = 5$ is the same. With higher confidence level, the advantage in using C-Tz statistics increases both in efficiency and noise. The advantage in using the C-Tz statistics is very large if the jumps in the simulated trajectory are consecutive. In this case, the efficiency of the z test at the 99.99% confidence level is just 42.4%, while the corresponding efficiency of the C-Tz test is 93.1%. We conclude that, on our simulations, there is a clear advantage in using the C-Tz statistics.

The fact that the z statistics perform very well on continuous trajectories is not surprising, and is in line with the results of Huang and Tauchen (2005); what is surprising is the high power of z-tests in detecting jumps, since the z statistics is based on quantities which are poorly estimated, especially the quarticity.

	Panel A No jumps				Panel B Single jump				Panel C Two consecutive jumps			
	50%	95%	99%	99.99%	50%	95%	99%	99.99%	50%	95%	99%	99.99%
z	53.0	5.7	1.4	0.1	93.4	81.2	77.6	68.6	98.1	79.1	64.4	42.4
z_1	53.0	8.9	3.3	0.4	93.4	82.4	79.8	73.5	98.1	89.7	85.9	76.2
z_2	53.0	8.3	2.8	0.2	93.4	82.3	79.4	72.2	98.1	81.4	72.5	55.6
Tz	63.7	21.7	11.0	1.6	96.0	88.8	85.8	79.0	99.6	98.8	98.4	96.9
Tz_1	63.7	24.8	16.4	4.7	96.0	89.7	87.3	81.3	99.6	98.8	98.7	97.7
Tz_2	63.7	24.4	15.3	4.2	96.0	89.6	87.3	81.4	99.6	98.8	98.6	97.7
C-Tz	54.0	6.0	1.6	0.1	93.7	83.6	80.6	74.6	99.2	97.3	96.3	93.1
$\mathsf{C\text{-}Tz}_1$	54.0	8.7	3.1	0.3	93.7	84.5	81.8	77.1	99.2	97.4	96.7	95.7
$\mathsf{C\text{-}Tz}_2$	54.0	8.7	3.1	0.3	93.7	84.3	81.7	76.9	99.2	97.5	96.4	95.1

Table 3: Percentage of detected jumps in the case of trajectories with a no jumps (Panel A), a single jump per day (Panel B), and two consecutive jumps per day (Panel C), for different significance levels. The C-Tz statistics are computed with $c_\vartheta = 3$.

The motivation for this result is that the bias is larger with large jumps; but when the jumps are very large, they are also easier to detect. Thus, even if corrected threshold multipower estimators provide much less biased estimates of integrals of power volatility, the advantage in using C-Tz statistics with respect to the corresponding z statistics is not as large as one would expect from Table 2. This reasoning is instead completely reversed if the jumps are consecutive; in this case, the quantities used to compute the z statistics are hugely biased, see Table 2, and this reduces its efficiency dramatically.

Again, we can check the sensitivity of C-Tz tests with respect to the choice of the parameter c_ϑ. Figure 3 shows a direct comparison between threshold estimators in the case of a single jump in every trajectory. We observe that the C-Tz test has more power than the z test and it is reasonably robust to the choice of c_ϑ.

Summarizing our result on simulated experiments, we conclude that:

1. When measuring integrated power volatility in the presence of jumps, multipower variation is largely upward biased, while threshold-based estimator are slightly downward biased.
2. When measuring integrated power volatility, threshold multipower estimators are nearly insensitive to the choice of the threshold for $c_\vartheta \geq 3$ while threshold power estimators are more sensitive to this choice.

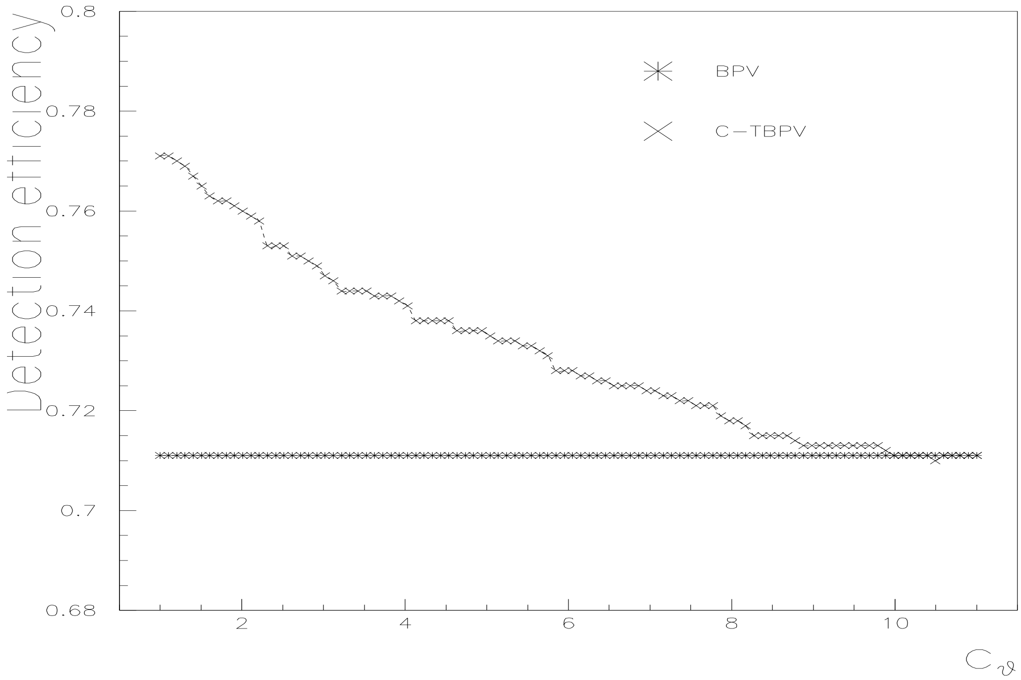

Figure 3: Jump detection efficiency for the model (3.1) in the presence of a single jump, as a function of the threshold parameter c_ϑ.

3. When testing for the presence of jumps, corrected threshold multipower estimators yield a significant advantage with respect to multipower variation based tests.

The latter conclusion will be enforced by results in the next section. Thus, in our empirical analysis we will use threshold bipower variation as an estimator of integrated volatility, and C-Tz statistics as jump detector.

4 Detecting jumps in data with jumps

In this Section, to further show the pitfalls which can be encountered in detecting jumps, we test the z statistics and the newly proposed C-Tz statistics on data sets of *daily* observations which are well known to display jumps. We use formulas (A.7) and (2.24) to define the statistics.

The first set is represented by the time series of daily electricity prices. One basic feature of daily electricity prices is to display very pronounced spikes, basically due to power shortages, see e.g Knittel and Roberts (2005); Geman and Roncoroni (2006) and references therein. We compute the jump detecting statistics on the whole time series of daily logarithmic returns of eight electricity markets, both European and American.

These data sets display very pronounced jumps (the time series are plotted in the web appendix).

We find that the z statistics is not able to detect jumps in the electricity time series. Jumps are detected only in Pennsylvania; in the other markets, the null of a continuous trajectory is not rejected, which is completely untenable for electricity prices. This disappointing results can be explained by the fact that most of the jumps in electricity returns are consecutive. After a spike, the price immediately reverts to its original level, generating a second large jump. This completely messes up the measures based on multipower variation, which hinges on the fact that, after a jump, there is a small return vanishing to zero in the limit.

To better point out this effect, we also compute the z test with staggered measures of multipower variation. Staggered bipower variation has been suggested by Huang and Tauchen (2005) and ABD to accommodate microstructure effects in high-frequency time series by breaking the correlation in the bid-ask noise. The staggered multipower variation is defined as:

$$\mathsf{stag\text{-}MPV}_\delta(X)_t^{[\gamma_1,\dots,\gamma_M]} = \frac{M}{M-2}\delta^{1-\frac{1}{2}(\gamma_1+\ldots+\gamma_M)} \sum_{j=M+1}^{[T/\delta]} \prod_{k=1}^{M} |\Delta_{j-2(k-1)}X|^{\gamma_k} \tag{4.1}$$

The tests based on staggered multipower variation perform indeed much better, as shown in Table 4. However, they would have poor power in detecting jumps in the case of three consecutive jumps, or in the case of “staggered” jumps. All these cases may be relevant in high frequency data, as those analyzed in the forthcoming sections.

A much better performance is instead obtained by the corrected test, as suggested by the theoretical properties of threshold multipower variation. When $c_\vartheta = 3$ we detect jumps in all the electricity markets, with the exception of Ohio, which has a kurtosis closer to the Normal value[4]; with $c_\vartheta = 5$ the test has less power, and difficulties are found in time series with moderately high kurtosis (France, The Netherlands). This is not surprising, since in the limit $c_\vartheta \to \infty$ the $\mathsf{C\text{-}Tz}$ test is identical to the z test. Thus, with an higher c_ϑ, the corrected test gets closer to the z test.

We also analyze the time series of daily interest rate differences. Interest rates are also well known to display jumps, especially in presence of macroeconomic announcements, see e.g. Das (2002); Johannes (2004). Moreover, it is well known that shorter maturity instruments display more jumps, because of liquidity reasons. We then analyze the time series of daily observations for both the 7-day Eurodollar deposit rate and the 3-month Treasury Bill rate, in the period starting in June 1973 and ending in February, 1995, for a total of 5,505 observations. The 7-day time series displays many jumps, and most of them are consecutive; an extensive discussion of the jump properties of these two data sets can be found in Renò et al. (2006).

The results are again shown in Table 4. On the 3-months time series, where jumps are rarely consecutive, all the considered statistics detect the jumps. On the 7-days time series jumps are most pronounced; but the z

[4]It is important to remark that high kurtosis does not imply presence of jumps; the presence of jumps in electricity market is witnessed in the related empirical literature, if not evident by visual inspection of prices.

Market	period	Kurtosis	z	stag-z	C-Tz	
					$c_\vartheta = 3$	$c_\vartheta = 5$
		Electricity markets				
Germany	1 Jan 01 → 25 Jul 05	29.9	1.59	8.63	10.68	7.82
France	27 Nov 01 → 30 Jul 05	12.6	0.98	8.19	4.37	1.72
Spain	1 Jan 01 → 26 May 05	10.5	0.83	5.79	5.00	2.99
The Netherlands	1 Jan 01 → 10 Aug 05	11.6	1.01	6.00	3.20	1.63
California	2 Jan 02 → 12 Feb 07	52.9	-0.11	6.77	10.57	6.76
Texas	2 Jan 02 → 19 Mar 07	27.4	-0.91	1.61	8.18	4.68
Ohio	3 Jan 01 → 11 May 07	4.9	0.51	2.83	1.41	1.28
Pennsylvania	2 Jan 01 → 9 May 07	12.4	3.45	4.87	5.79	4.53
		Interest rates				
7-days	1 Jun 1973 - 24 Feb 1995	30.8	2.02	11.51	15.39	7.52
3-months	1 Jun 1973 - 24 Feb 1995	18.6	6.90	6.22	8.73	7.04

Table 4: Jump detection statistics on daily electricity price returns in different markets, and daily interest rate differences for two maturities.

statistics is not able to detect them. The fact that there are many consecutive jumps is signaled by the high value of the staggered z statistics. The corrected C-Tz statistics also detect the jumps; with a value much higher than that of the 3-months time series, in accordance with the evidence that jumps are more frequent and more pronounced in the 7-day time series. The C-Tz statistics is lower at $c_\vartheta = 5$, since it becomes closer to the z statistics as c_ϑ increases.

Concluding, consistently with the theory in the previous section, we find that the z statistics can be very misleading in some situations, e.g. consecutive jumps. On the contrary, the C-Tz statistics is more efficient in detecting jumps, especially when consecutive.

5 The forecasting model

Empirical evidence on strong temporal dependence of realized volatility has been already found for instance in Andersen, Bollerslev, Diebold and Labys (2001) and Andersen, Bollerslev, Diebold and Ebens (2001). This evidence, together with our empirical results reported below, suggests that realized volatility series should be described by long-memory type of models, see Andersen et al. (2003); Banerjee and Urga (2005); McAleer and Medeiros (2006).

Recently, Corsi (2004) and Corsi et al. (2008) introduced a class of time series models called Heterogeneous Auto-Regressive (HAR) models that successfully achieves the purpose of modeling the long memory behavior of financial variables in a simple and parsimonious way.

Inspired by the HARCH of Muller et al. (1997) where the conditional variance is function of squared returns over many horizons, Corsi (2004) proposed a stochastic additive cascade of three different realized volatility components corresponding to the three main different time horizons operating in the market (daily, weekly and monthly). This stochastic volatility cascade leads to simple AR-type models in the realized volatility with the feature of considering realized volatilities defined over different time horizons (the HAR models). Although the HAR model does not formally belong to the class of long-memory models, it is able to reproduce a memory decay indistinguishable from that observed in empirical data.

To explicitly define the HAR models, and to extend them to disentangle the different contributions of continuous and discontinuous quadratic variation, let first define the average multiperiod realized variation i.e. a realized variation measure aggregated over longer horizons and normalized to the daily scale:

$$\mathsf{RV}_{t-h:t} = \frac{1}{h}(\mathsf{RV}_{t-h} + \mathsf{RV}_{t-h+1} + ... + \mathsf{RV}_{t}) \tag{5.1}$$

To keep the HAR model simple and intuitive we use three aggregation frequencies: daily ($h = 1$), weekly ($h = 5$) and monthly ($h = 22$). The standard HAR model can then be written as

$$\mathsf{RV}_{t+1:t+h} \quad = \quad \beta_0 + \beta_d \,\mathsf{RV}_t + \beta_w \,\mathsf{RV}_{t-5:t} + \beta_m \,\mathsf{RV}_{t-22:t} + \varepsilon_t \tag{5.2}$$

where ε_t is a standard IID noise and where $h = 1$ forecasts daily volatility, $h = 5$ forecasts weekly volatility and $h = 22$ forecasts monthly volatility. This model has been widely employed in recent applications; see for example Forsberg and Ghysels (2007); Maheu and McCurdy (2006); Clements et al. (2008).

In what follows, the HAR model is extended to directly incorporate the different contribution of jumps. ABD first added a jump component as

$$\mathsf{RV}_{t+1:t+h} \quad = \quad \beta_0 + \beta_d \,\mathsf{RV}_t + \beta_w \,\mathsf{RV}_{t-5:t} + \beta_m \,\mathsf{RV}_{t-22:t} + \beta_j \widehat{J}_t + \varepsilon_t \tag{5.3}$$

where $\widehat{J}_t$ is estimated as

$$\hat{J}_t = I_{t,J} \cdot (\mathsf{RV}_t - \mathsf{BPV}_t)^+ \,, \tag{5.4}$$

$x^+ = max(x, 0)$, and $I_{t,J}$ is an indicator function which is equal to 1 if jumps are detected on day t, and equal to 0 elsewhere. In their study, ABD use $I_{t,J} = I_{\{z_t > \Phi_\alpha\}}$, with z_t given by (A.7) and Φ_α is the quantile function of the Normal distribution at confidence level α. Clearly, different tests can be adopted, as for example (A.8) and (A.9).

ABD estimated β_j in equation (5.3) to be negative. Now, note that the total realized variance can be decomposed in its continuous (C_t) and jump (J_t) component, (i.e. $\mathsf{RV}_t = C_t + J_t$) and that RV_t enters both the weekly ($\mathsf{RV}_{t-5:t}$) and monthly ($\mathsf{RV}_{t-22:t}$) measures; as a consequence the jump component J_t is present in all the regressors of model (5.3) making the interpretation of the impact of J_t non trivial. In order to correctly isolate the impact of the jump component J_t, an estimate of the continuous part should be employed as dependent variable in place of the realized variance. A natural estimate of the continuous component is simply the difference between the realized variance and the estimated significant jump. Estimating the jump component as in equation (5.4) and denoting the corresponding continuous part as

$$\widehat{C_t} = \mathsf{RV}_t - \widehat{J}_t \tag{5.5}$$

we get the following HAR-CJ model, similar to that used by ABD in their forecasting analysis[5]:

$$\mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d \widehat{C}_t + \beta_w \widehat{C}_{t-5:t} + \beta_m \widehat{C}_{t-22:t} + \beta_j \widehat{J}_t + \varepsilon_t \tag{5.6}$$

where $\widehat{C}_{t-5:t}$ and $\widehat{C}_{t-22:t}$ are, analogously to the realized volatility measures, the weekly and monthly average multiperiod aggregation of the daily continuous component $\widehat{C_t}$. Using model (5.6), ABD still estimate β_j to be not significant, and the same conclusion has been reached by Forsberg and Ghysels (2007), Giot and Laurent (2007) and Busch et al. (2006); see also the analysis of Ghysels et al. (2006).

In the light of the above sections, it is natural to estimate the jump component using threshold bipower variation instead. Moreover, since we introduced threshold multipower variation, we have more flexibility in choosing the jump detector $I_{t,J}$ and the measures of C_t and J_t in days in which $I_{t,J} = 1$. The motivation for using an alternative bipower variation estimator is that in the measures (5.4) and (5.5) an unbiased estimate of $[X^c]$ in the presence of jumps is needed. Now, Sections 3 and 4 show that bipower variation is a largely biased measure of $[X^c]$ in days where jumps are present. For this reason, we use TBPV in days in which jumps are detected.

Formally, we define the HAR-TCJ model as:

$$\mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d \widehat{TC}_t + \beta_w \widehat{TC}_{t-5:t} + \beta_m \widehat{TC}_{t-22:t} + \beta_j \widehat{TJ}_t + \varepsilon_t \tag{5.7}$$

where we employ the threshold bipower variation measure to estimate the jump component

$$\widehat{TJ}_t = I_{t,J} \cdot (\mathsf{RV}_t - \mathsf{TBPV}_t)^+ \tag{5.8}$$

[5] ABD also consider weekly and monthly aggregation of the jump component.

and the corresponding continuous part $\widehat{TC}_t = \mathsf{RV}_t - \widehat{TJ}_t$, which is equal to RV_t if $I_{t,J} = 0$ and to TBPV_t if $I_{t,J} = 1$.

Its square-root and logarithmic counterparts will read:

$$\mathsf{RV}_{t+1:t+h}^{\frac{1}{2}} = \beta_0 + \beta_d \widehat{TC}_t^{\frac{1}{2}} + \beta_w \widehat{TC}_{t-5:t}^{\frac{1}{2}} + \beta_m \widehat{TC}_{t-22:t}^{\frac{1}{2}} + \beta_j \widehat{TJ}_t^{\frac{1}{2}} + \varepsilon_t \tag{5.9}$$

and

$$\log \mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d \log \widehat{TC}_t + \beta_w \log \widehat{TC}_{t-5:t} + \beta_m \log \widehat{TC}_{t-22:t} + \beta_j \log\left(\widehat{TJ}_t + 1\right) + \varepsilon_t \tag{5.10}$$

and the same transformations[6] will be estimated for model (5.2) and (5.6).

To evaluate the forecasting performance of the different models, we use the R^2 of Mincer-Zarnowitz forecasting regressions, as well as the (relative) RMSE in predicting the square root of RV, defined as:

$$RMSE = \frac{1}{T}\left[\sum_{t=1}^{T}\left(\frac{\mathsf{RV}_t^{\frac{1}{2}} - \widehat{\mathsf{RV}}_t^{\frac{1}{2}}}{\mathsf{RV}_t^{\frac{1}{2}}}\right)^2\right]^{\frac{1}{2}} \tag{5.11}$$

where $\mathsf{RV}_t^{\frac{1}{2}}$ is the measured value of realized volatility and $\widehat{\mathsf{RV}}_t^{\frac{1}{2}}$ is the predicted value implicit in the model[7]. We use the square root of RV since this is the value mostly employed in risk management applications.

6 Empirical Analysis

Our data set covers a long time span of almost 15 years of high frequency data for the S&P 500 futures and US Treasury Bond with maturity 30 years, and nearly 5 years of 6 individual stocks. The purpose of this section is mainly to analyze the impact of jumps on future volatility when threshold bipower variation is employed as a measure of jumps. We will focus not only on the impact of jumps on future realized volatility, but also on the performance of models which explicitly incorporate jumps in forecasting volatility.

All the analysis presented in this section is based on measures of threshold multipower variation with a value $c_\vartheta = 3$, and using the $\mathsf{C\text{-}Tz}$ statistics to detect jumps. Our tables are built at confidence level $\alpha = 99.9\%$ but the most interesting quantities will be computed and plotted for different values of α as well. Further results with $c_\vartheta = 4, 5$ can be found in a companion web appendix.[8]

[6] We remark that in the logarithmic model the term $\log\left(\widehat{TJ}_t + 1\right)$ makes the estimates of the parameters $\beta_d, \beta_w, \beta_m, \beta_j$ not invariant to time scaling. To get approximate invariance, it is advisable to choose a time unit such that $\widehat{TJ}_t >> 1$. For this reason, in what follow we use annualized quantities.

[7] Patton (2006) suggest to use the absolute MSE as a proper loss function, see also the related discussion in Forsberg and Ghysels (2007). However, the ranking of models made in section 6 does not change if we use MSE instead of the relative $RMSE$ (5.11), which we find more intuitive since it gives immediately the relative forecasting error.

[8] The web appendix can be downloaded at `http://www.econ-pol.unisi.it/~reno`, in the Research section.

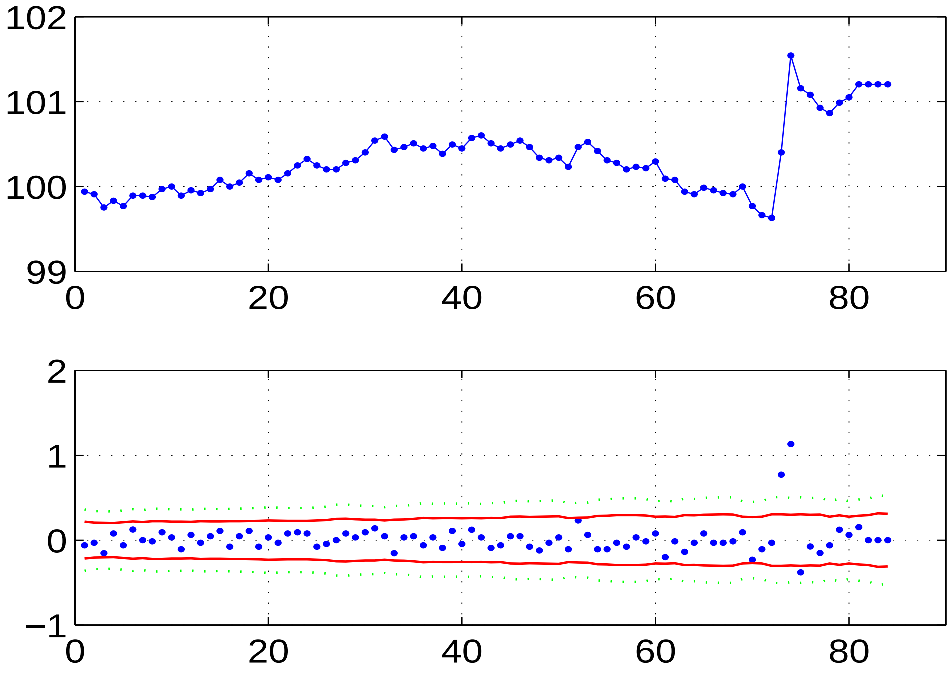

Figure 4: Rescaled time series (top) and 5-minutes logarithmic returns (bottom) of the S&P500 on 12 April 1990. The solid and the dotted line are our estimated threshold with $c_\vartheta = 3$ and $c_\vartheta = 5$ respectively. The jump statistics are $z = -0.2545$, $\mathsf{C\text{-}Tz} = 4.5055$ with $c_\vartheta = 3$, $\mathsf{C\text{-}Tz} = 4.4745$ with $c_\vartheta = 5$.

6.1 Stock index futures S&P500 data

The first data set we analyze is the S&P500 futures time series. We dispose of all high-frequency transactions from January 1990 to December 2004 (3,736 days). In order to mitigate the impact of microstructure effects on our estimates, we choose, as in ABD, a sampling frequency of $\delta = 5$ minutes, corresponding to 84 returns per day.

Figure 4 is an example in which using the C-Tz statistics is effective. It displays the S&P 500 time series on one specific day, in which there is an evident jump. However, in this day, the z statistics, which is based on multipower variation, is negative and does not reveal a jump at any reasonable significance level; while the C-Tz statistics, which is based on threshold multipower variation, does reveal a very significant jump. Our interpretation for this day is that, since the jump appears in the form of two consecutive and very large returns, this creates a huge bias (especially in the quarticity estimates) which makes the z statistics very noisy. This bias is instead completely removed by the corrected threshold estimators. Other examples of this kind are shown in the web appendix.

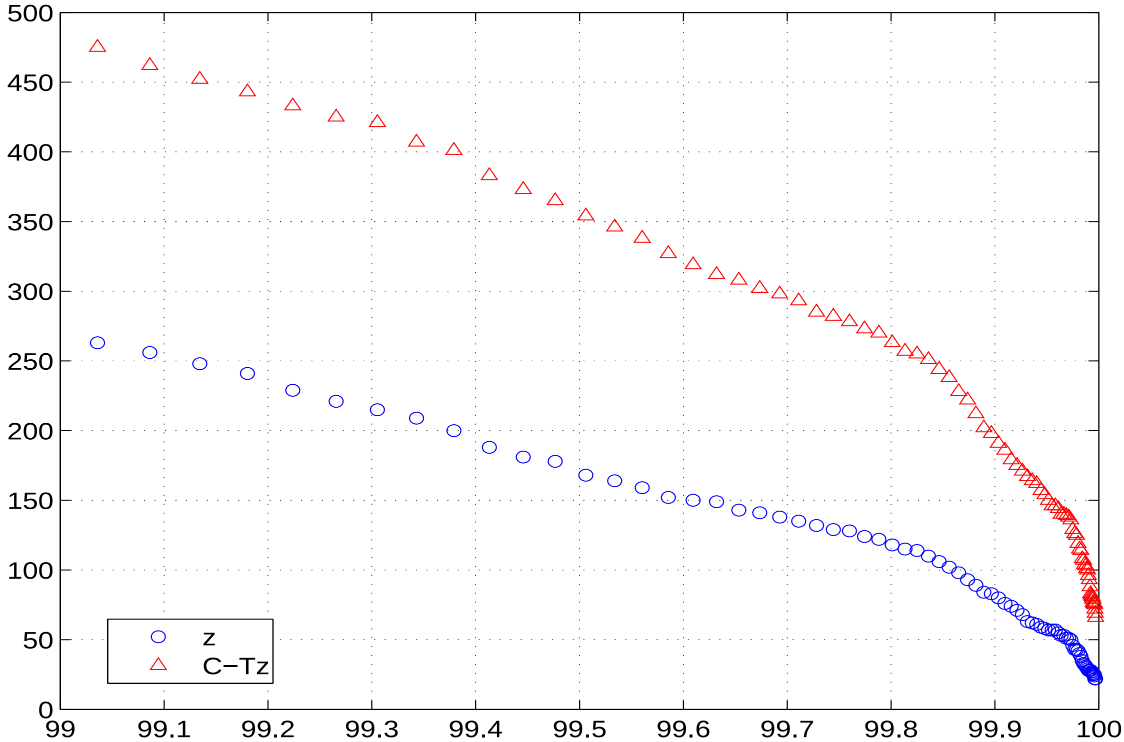

Figure 5: Number of days which contain jumps in the S&P500 sample obtained with the C-Tz statistics (2.24), as a function of the confidence level α. The total number of days is $3,736$.

Figure 5 shows the number of jumps in this sample, as detected by the condition C-Tz $> \Phi_\alpha$ and $z > \Phi_\alpha$ as a function of α. We see that with the statistics based on threshold multipower variation, we get an higher number of jumps.

We estimate model (5.7), which we compare with model (5.6). We also estimate the standard HAR model (5.2) as comparison with a model with no separation between continuous and discontinuous component. Results are reported in Table 5, where all jumps have been estimated with the C-Tz statistics.

Results are unambiguous. When the jump component is measured by means of bipower variation, as in the HAR-CJ model, its coefficient is significantly negative for the square model and insignificant for the log and square root model in explaining future volatility. This surprising result is at odd, in our opinion, with the economic intuition which would suggest an increase in volatility after a jump in the price process (especially if large and negative). Moreover, this result is even more puzzling, given that the unconditional correlations between realized variation and jumps lagged by one day is significantly positive and around 20% for the variances, 30% for the volatilities and 25% for the log volatilities.

The explanation for this result is that the continuous component $\widehat{C_t}$ estimated using bipower variation is still contaminated by the jump component and hence, as for the case with the realized variance in model (5.3), the impact of jumps is also passing through the positive coefficients of the other regressors.

When instead the jump component is measured by means of threshold bipower variation, the coefficients β_j are positive, and highly significant in explaining future volatility for the square root and log model. Most importantly, the HAR-TCJ model yields an higher R^2 and a lower $RMSE$, thus showing a better forecasting power. To better understand this point, we divide the sample in days immediately following the occurrence of a jump, and days which follow days with no jumps. On these two samples we compute the R^2 and $RMSE$ statistics separately, denoting them by $J-R^2, J-RMSE$ and $C-R^2, C-RMSE$, respectively. The results in Table 5 show that the TCJ model largely improves the forecasting power on realized volatility in days immediately following a jump, and it is still slightly more performing in days which do not follow a jump. Our interpretation of this result is that, since we are better measuring the jump component, we are also removing noise from the continuous component in the explanatory variables; and thus, we also get slightly better results on days in which there were no jumps before.

Our findings are quite robust to the chosen confidence interval. Figure 6 displays the most important quantities as a function of the confidence interval α. It shows that the jump component of the HAR-TCJ model, as measured by the $t-$statistics of the coefficient β_j, is always positive for all models and highly significant for square root and log transformations; while the jump component of the HAR-CJ model is mainly significantly negative or not significant. Importantly, it shows that the HAR-TCJ model provides superior forecasts when measured in terms of R^2 and the $RMSE$, irrespective of the confidence level used and model employed.

Figure 7 summarizes our findings. It reports the mean square error in forecasting daily realized volatility ($h = 1$) for all the considered models, computed on the full sample (top) and only on days subsequent the occurrence of a jump (bottom). The figure shows that, using the $RMSE$ metrics, in the full sample the ranking depends on the specification: logarithmic models outperform the square root models, which in turn outperform the square models. However, for all the three specifications the best performance is obtained by the HAR-TCJ models which employ threshold bipower variation.

Most importantly, in days following a jump the ranking based on $RMSE$ depends on the measure adopted for continuous variation. Indeed, the HAR-TCJ models provide the smaller $RMSE$, either in the logarithmic, square root or square specification, and their performance does not decline with increasing confidence level.

We also estimate the forecasting models using weekly realized volatility and monthly realized volatility as dependent variables. Results are shown in Tables 6 and 7, respectively. When forecasting weekly and monthly volatility, the β_j of the HAR-CJ model tends to be negative, sometimes significantly. Instead, for the HAR-TCJ model, the β_j are largely positive and significant in the square root and log model, and insignificant in the square model. Again, the R^2 and the $RMSE$ confirm, in days following a jump, the

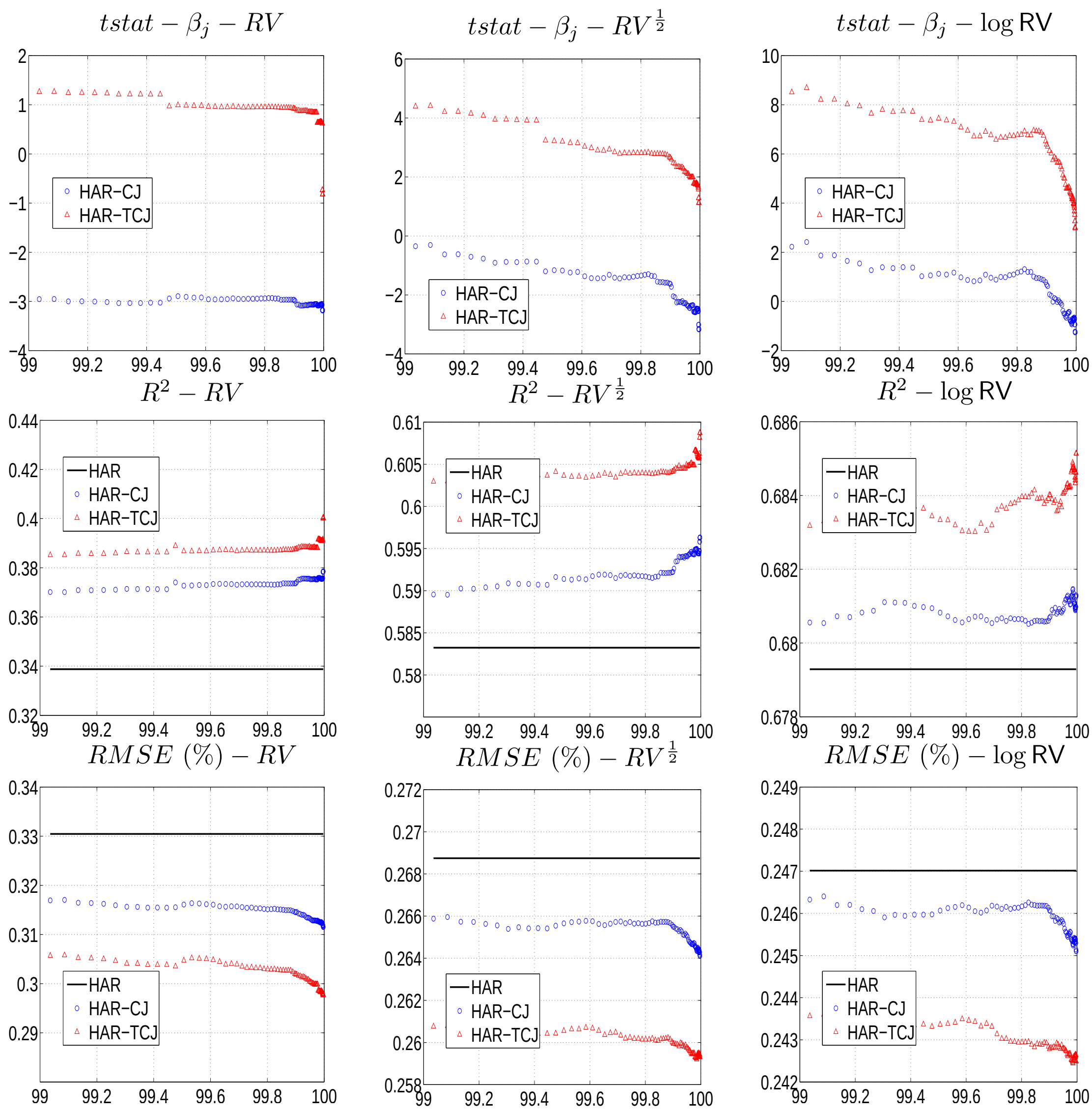

Figure 6: Reports the t statistics of the coefficient β_j, the R^2 and the $RMSE$ for the three models estimated on S&P 500 data for daily forecasting, for both the HAR-CJ and HAR-TCJ versions, as a function of the confidence level used for detecting jumps with the C-Tz statistics.

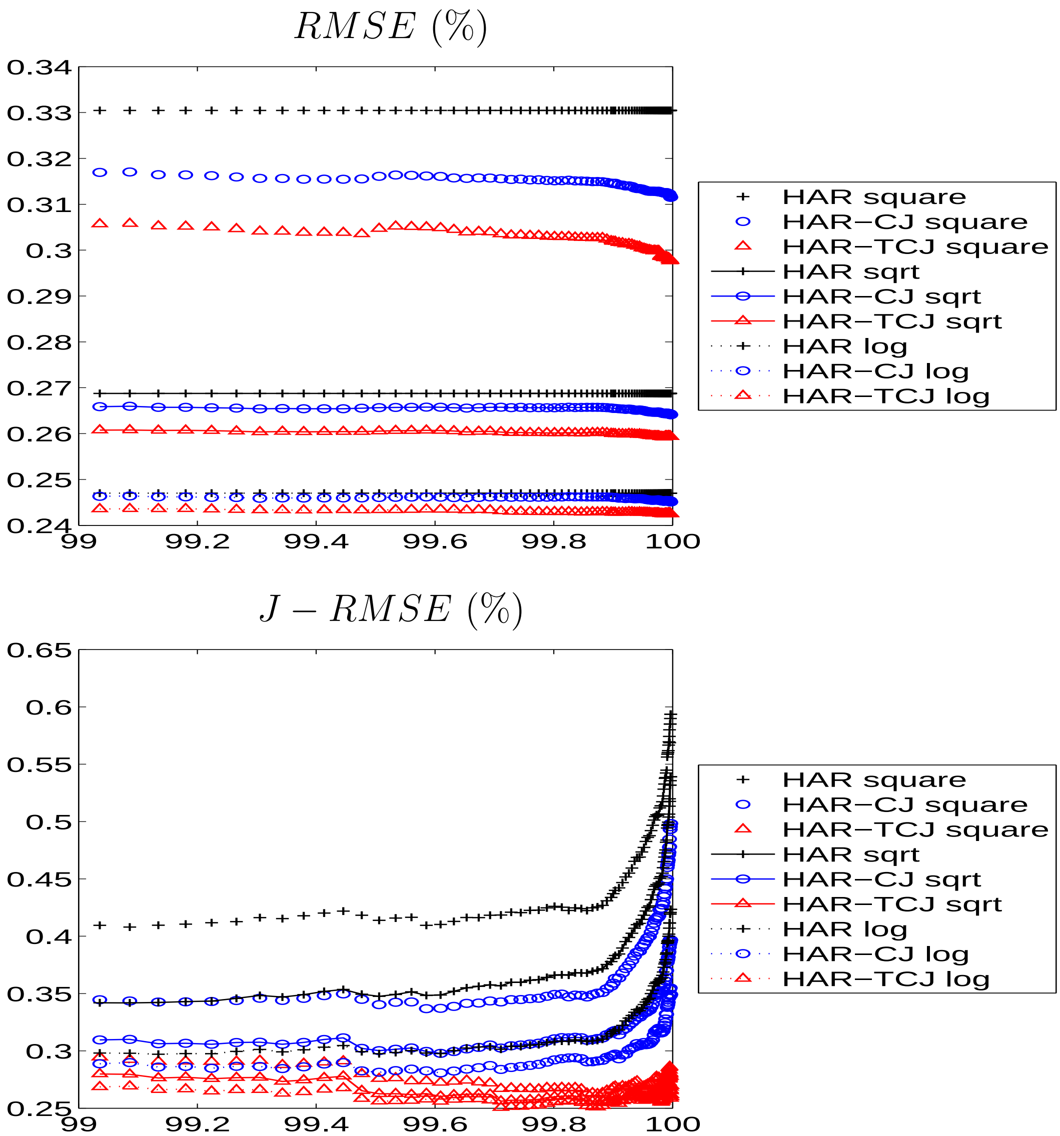

Figure 7: Top: $RMSE$ of daily forecasts as a function of the confidence intervals, for all the considered models. Bottom: $J - RMSE$, that is the mean square error restricted to days immediately after a jump.

better forecasting ability of the HAR-TCJ model, which is not worse than HAR-CJ in days not following a jump. Thus, the forecasting ability of the jump component, when it is measured by means of threshold bipower variation, extends to a time span of at least one month.

The analysis with higher values of c_ϑ, shown in the web appendix, reveals that the β_j coefficient of the TCJ specification is mildly significant for $c_\vartheta = 4$ and not significant for $c_\vartheta = 5$. This is not surprising, since as we increase c_ϑ we get closer to the results obtained with bipower variation. We also estimate the HAR-CJ model using the jumps detected via the z statistics (A.7), as in ABD, and compare it with the HAR-TCJ model estimated with the jumps detected via the C-Tz statistics (2.24). The results, omitted here for brevity and shown in the web appendix, indicate that in this case the difference between the two models is even higher and that the R^2 and $RMSE$ of the HAR-CJ model are drastically worse for higher α, signaling the difficulty of bipower variation based test to identify larger jumps. When using the same jumps detected with the less performing z statistics, results are qualitatively unchanged, but with milder significance of jumps and with the two models performing more closely for the highest confidence levels.

6.2 Individual stocks

We analyze a sample of six individual stocks, chosen among the most liquid stocks of S&P500. The stocks are Alcoa (aa), Citigroup (c), Intel (intc), Microsof (msft), Pfeizer (pfe) and Exxon-Mobil (xom). Our sample starts on 2 January 2001 and ends on 30 December 2005, containing $1,256$ days per stock. Since these stocks are traded very actively, we still use a sampling frequency of $\delta = 5$ minutes, corresponding to 78 returns per day.

To save space, we focus on the most important quantities (the significance of the jump and the R^2 and $RMSE$ of the forecasting model both on the whole sample and conditioned on days after the occurrence of a jump), which are reported in Table 6.2 for the square root model and Table 6.2 for the logarithmic model. We report results for $\alpha = 99.9\%$ and $\alpha = 99.99\%$.

The results are still clearly cut, even if the sample size of individual stocks is reduced by more than one third with respect to S&P500 data. Jumps have a substantial impact in determining future volatility, when this effect is measured by means of threshold multipower variation. The t statistics of the β_j coefficient is always larger for the HAR-TCJ model than for the HAR-CJ model, and mostly significant both for the square root and the logarithmic model. On the whole sample, the performance, measured in terms of R^2 and $RMSE$, of the two models is practically the same, but conditioned on the occurrence of a jump, there is a clear advantage in using the HAR-TCJ model, especially in the square root specification.

Thus, the results obtained on the S&P500 portfolio are replicated on its most liquid constituents, indicating that the impact of jumps on future volatility is not peculiar to the S&P500 returns considered in the previous Section, and suggesting that it may simply come from aggregation, at the portfolio level, of the same effect at the individual stock level.

Daily S&P500 Regression (C-Tz statistics)

$$\text{HAR:}\quad \mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d\,\mathsf{RV}_t + \beta_w\,\mathsf{RV}_{t-5:t} + \beta_m\,\mathsf{RV}_{t-22:t} + \varepsilon_t$$
$$\text{HAR-CJ:}\quad \mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d\widehat{C}_t + \beta_w\widehat{C}_{t-5:t} + \beta_m\widehat{C}_{t-22:t} + \beta_j\widehat{J}_t + \varepsilon_t$$
$$\text{HAR-TCJ:}\quad \mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d\widehat{TC}_t + \beta_w\widehat{TC}_{t-5:t} + \beta_m\widehat{TC}_{t-22:t} + \beta_j\widehat{TJ}_t + \varepsilon_t$$

	RV_{t+1}			$\mathsf{RV}_{t+1}^{1/2}$			$log\,\mathsf{RV}_{t+1}$		
	HAR	HAR-CJ	HAR-TCJ	HAR	HAR-CJ	HAR-TCJ	HAR	HAR-CJ	HAR-TCJ
β_0	34.200	26.762	23.259	0.981	0.901	0.767	0.252	0.272	0.284
	(3.771)	(3.363)	(2.753)	(3.951)	(3.795)	(3.298)	(4.457)	(4.866)	(5.168)
β_d	0.220	0.378	0.420	0.323	0.371	0.421	0.334	0.336	0.356
	(2.329)	(5.736)	(6.170)	(6.347)	(8.958)	(11.954)	(13.138)	(13.140)	(14.829)
β_w	0.321	0.263	0.298	0.336	0.317	0.307	0.358	0.356	0.341
	(3.821)	(3.157)	(2.609)	(6.075)	(5.811)	(5.427)	(9.463)	(9.398)	(9.421)
β_m	0.313	0.288	0.253	0.269	0.260	0.238	0.257	0.257	0.253
	(4.817)	(4.786)	(3.852)	(6.842)	(6.725)	(6.103)	(8.873)	(8.959)	(9.067)
β_j		-0.581	0.045		-0.101	0.096		0.007	0.055
		(-2.968)	(0.925)		(-1.626)	(2.653)		(0.683)	(6.384)
R^2	0.339	0.374	0.387	0.583	0.592	0.604	0.679	0.681	0.684
MSE	0.330	0.315	0.302	0.269	0.266	0.260	0.247	0.246	0.243
J-R^2	0.148	0.196	0.306	0.373	0.385	0.480	0.607	0.607	0.643
J-MSE	0.437	0.359	0.268	0.381	0.317	0.261	0.317	0.296	0.256
C-R^2	0.375	0.394	0.399	0.609	0.613	0.616	0.686	0.687	0.688
C-MSE	0.322	0.311	0.304	0.259	0.262	0.260	0.242	0.242	0.242

Table 5: OLS estimate for daily ($h = 1$) HAR, HAR-CJ and HAR-TCJ volatility forecast regressions for S&P500 futures from January 1990 to December 2004 (3,736 observations). The significant daily jump are computed using a critical value of $\alpha = 99.9\%$ and the C-Tz statistics computed with $c_\vartheta = 3$. Reported in parenthesis are the t-statistics based on Newey-West correction with order 5.

Weekly S&P500 Regression

$$\text{HAR:} \quad \mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d \, \mathsf{RV}_t + \beta_w \, \mathsf{RV}_{t-5:t} + \beta_m \, \mathsf{RV}_{t-22:t} + \varepsilon_t$$
$$\text{HAR-CJ:} \quad \mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d \widehat{C}_t + \beta_w \widehat{C}_{t-5:t} + \beta_m \widehat{C}_{t-22:t} + \beta_j \widehat{J}_t + \varepsilon_t$$
$$\text{HAR-TCJ:} \quad \mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d \widehat{TC}_t + \beta_w \widehat{TC}_{t-5:t} + \beta_m \widehat{TC}_{t-22:t} + \beta_j \widehat{TJ}_t + \varepsilon_t$$

	RV_{t+1}			$\mathsf{RV}_{t+1}^{1/2}$			$log\,\mathsf{RV}_{t+1}$		
	HAR	HAR-CJ	HAR-TCJ	HAR	HAR-CJ	HAR-TCJ	HAR	HAR-CJ	HAR-TCJ
β_0	47.231	41.199	37.791	1.532	1.457	1.340	0.403	0.422	0.435
	(4.324)	(3.976)	(3.375)	(4.306)	(4.223)	(3.937)	(4.790)	(5.066)	(5.332)
β_d	0.097	0.190	0.210	0.176	0.213	0.244	0.205	0.210	0.229
	(1.892)	(4.858)	(4.402)	(5.360)	(7.925)	(9.392)	(11.742)	(12.129)	(13.815)
β_w	0.367	0.351	0.392	0.368	0.355	0.352	0.358	0.350	0.330
	(4.676)	(4.298)	(3.564)	(6.145)	(5.995)	(5.858)	(8.179)	(8.081)	(8.001)
β_m	0.335	0.320	0.305	0.344	0.339	0.330	0.357	0.359	0.362
	(4.901)	(4.375)	(3.707)	(6.390)	(6.203)	(6.060)	(8.601)	(8.779)	(9.205)
β_j		-0.394	0.007		-0.105	0.040		-0.005	0.031
		(-2.570)	(0.378)		(-2.425)	(2.465)		(-0.689)	(6.190)
R^2	0.499	0.534	0.554	0.690	0.700	0.709	0.768	0.770	0.772
MSE	0.273	0.260	0.252	0.205	0.202	0.199	0.187	0.186	0.185
$J\text{-}R^2$	0.475	0.464	0.445	0.649	0.639	0.632	0.733	0.732	0.729
J-MSE	0.277	0.267	0.265	0.217	0.218	0.219	0.209	0.209	0.210
$C\text{-}R^2$	0.506	0.545	0.568	0.695	0.706	0.716	0.771	0.773	0.776
C-MSE	0.273	0.259	0.251	0.205	0.201	0.198	0.186	0.184	0.183

Table 6: OLS estimate for weekly ($h = 5$) HAR, HAR-CJ and HAR-TCJ volatility forecast regressions for S&P500 futures from January 1990 to December 2004 (3,736 observations). The significant daily jump are computed using a critical value of $\alpha = 99.9\%$. Reported in parenthesis are the t-statistics based on Newey-West correction with order 10.

Monthly S&P500 Regression

$$\text{HAR:} \quad \mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d \, \mathsf{RV}_t + \beta_w \, \mathsf{RV}_{t-5:t} + \beta_m \, \mathsf{RV}_{t-22:t} + \varepsilon_t$$
$$\text{HAR-CJ:} \quad \mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d \widehat{C}_t + \beta_w \widehat{C}_{t-5:t} + \beta_m \widehat{C}_{t-22:t} + \beta_j \widehat{J}_t + \varepsilon_t$$
$$\text{HAR-TCJ:} \quad \mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d \widehat{TC}_t + \beta_w \widehat{TC}_{t-5:t} + \beta_m \widehat{TC}_{t-22:t} + \beta_j \widehat{TJ}_t + \varepsilon_t$$

	RV_{t+1}			$\mathsf{RV}_{t+1}^{1/2}$			$log\,\mathsf{RV}_{t+1}$		
	HAR	HAR-CJ	HAR-TCJ	HAR	HAR-CJ	HAR-TCJ	HAR	HAR-CJ	HAR-TCJ
β_0	78.416	73.455	70.663	2.872	2.797	2.688	0.753	0.767	0.774
	(5.896)	(5.623)	(4.948)	(5.872)	(5.947)	(5.862)	(5.269)	(5.469)	(5.680)
β_d	0.061	0.124	0.129	0.109	0.135	0.149	0.126	0.130	0.138
	(2.555)	(5.292)	(4.875)	(5.914)	(8.403)	(9.179)	(10.053)	(10.516)	(11.717)
β_w	0.219	0.206	0.242	0.279	0.272	0.279	0.266	0.261	0.250
	(4.274)	(3.742)	(4.344)	(5.547)	(5.354)	(5.366)	(6.098)	(6.019)	(5.848)
β_m	0.386	0.386	0.385	0.401	0.400	0.397	0.458	0.460	0.466
	(4.601)	(4.316)	(4.028)	(6.392)	(6.325)	(6.405)	(9.221)	(9.322)	(9.795)
β_j		-0.284	-0.002		-0.091	0.018		-0.010	0.016
		(-2.892)	(-0.114)		(-3.273)	(1.441)		(-1.738)	(3.660)
R^2	0.471	0.500	0.517	0.650	0.659	0.668	0.739	0.743	0.747
MSE	0.300	0.290	0.285	0.209	0.206	0.205	0.187	0.186	0.185
$J\text{-}R^2$	0.533	0.505	0.503	0.662	0.656	0.656	0.713	0.719	0.721
J-MSE	0.308	0.301	0.302	0.221	0.218	0.219	0.197	0.195	0.194
$C\text{-}R^2$	0.467	0.500	0.519	0.649	0.660	0.669	0.741	0.745	0.748
C-MSE	0.299	0.289	0.284	0.209	0.205	0.204	0.187	0.185	0.184

Table 7: OLS estimate for monthly ($h = 22$) HAR, HAR-CJ and HAR-TCJ volatility forecast regressions for S&P500 futures from January 1990 to December 2004 (3,736 observations). The significant daily jump are computed using a critical value of $\alpha = 99.9\%$. Reported in parenthesis are the t-statistics based on Newey-West correction with order 44.

$$\text{HAR:} \quad \mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d \,\mathsf{RV}_t + \beta_w \,\mathsf{RV}_{t-5:t} + \beta_m \,\mathsf{RV}_{t-22:t} + \varepsilon_t$$

$$\text{HAR-CJ:} \quad \mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d \hat{C}_t + \beta_w \hat{C}_{t-5:t} + \beta_m \hat{C}_{t-22:t} + \beta_j \hat{J}_t + \varepsilon_t$$

$$\text{HAR-TCJ:} \quad \mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d \hat{TC}_t + \beta_w \hat{TC}_{t-5:t} + \beta_m \hat{TC}_{t-22:t} + \beta_j \hat{TJ}_t + \varepsilon_t$$

Stock	α (%)	Jumps	β_j t-stat		R^2 (J-R^2)			$RMSE$ (J-$RMSE$)		
			HAR-CJ	HAR-TCJ	HAR	HAR-CJ	HAR-TCJ	HAR	HAR-CJ	HAR-TCJ
aa	99.9	121	1.086	3.684	0.597	0.601	0.596	0.247	0.245	0.245
					(0.534)	(0.559)	(0.569)	(0.314)	(0.292)	(0.289)
	99.99	68	-0.147	2.473	0.597	0.601	0.597	0.247	0.245	0.245
					(0.411)	(0.454)	(0.484)	(0.319)	(0.262)	(0.257)
c	99.9	105	-0.458	3.494	0.723	0.731	0.733	0.248	0.244	0.244
					(0.770)	(0.775)	(0.820)	(0.324)	(0.239)	(0.217)
	99.99	59	-0.937	3.139	0.723	0.732	0.731	0.248	0.244	0.244
					(0.830)	(0.854)	(0.856)	(0.361)	(0.231)	(0.219)
intc	99.9	78	2.706	4.925	0.774	0.774	0.773	0.226	0.226	0.226
					(0.828)	(0.833)	(0.837)	(0.227)	(0.218)	(0.214)
	99.99	43	1.483	3.461	0.774	0.774	0.774	0.226	0.226	0.225
					(0.786)	(0.794)	(0.803)	(0.257)	(0.235)	(0.230)
msft	99.9	92	0.611	2.721	0.748	0.749	0.749	0.242	0.242	0.241
					(0.752)	(0.753)	(0.760)	(0.311)	(0.279)	(0.265)
	99.99	48	0.661	2.567	0.748	0.748	0.748	0.242	0.242	0.241
					(0.652)	(0.643)	(0.674)	(0.363)	(0.332)	(0.311)
pfe	99.9	131	0.500	3.109	0.481	0.488	0.491	0.287	0.281	0.281
					(0.630)	(0.650)	(0.680)	(0.368)	(0.297)	(0.272)
	99.99	83	0.877	2.968	0.481	0.487	0.489	0.287	0.282	0.281
					(0.609)	(0.651)	(0.677)	(0.407)	(0.318)	(0.289)
xom	99.9	98	0.051	2.892	0.674	0.678	0.681	0.237	0.236	0.235
					(0.791)	(0.778)	(0.806)	(0.239)	(0.205)	(0.191)
	99.99	54	0.108	2.411	0.674	0.677	0.680	0.237	0.235	0.235
					(0.749)	(0.730)	(0.776)	(0.289)	(0.246)	(0.224)

Table 8: Reports number of jumps, t-stat on β_j, R^2, $RMSE$, $J - R^2$ and $J - RMSE$ for daily ($h = 1$) **square root** version of HAR, HAR-CJ and HAR-TCJ volatility forecast regressions on six single stocks . The significant daily jump are computed using a critical value of $\alpha = 0.999$ and $\alpha = 0.9999$ as reported, with the C-Tz statistics.

$$\text{HAR:} \quad \mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d \,\mathsf{RV}_t + \beta_w \,\mathsf{RV}_{t-5:t} + \beta_m \,\mathsf{RV}_{t-22:t} + \varepsilon_t$$

$$\text{HAR-CJ:} \quad \mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d \hat{C}_t + \beta_w \hat{C}_{t-5:t} + \beta_m \hat{C}_{t-22:t} + \beta_j \hat{J}_t + \varepsilon_t$$

$$\text{HAR-TCJ:} \quad \mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d \hat{TC}_t + \beta_w \hat{TC}_{t-5:t} + \beta_m \hat{TC}_{t-22:t} + \beta_j \hat{TJ}_t + \varepsilon_t$$

Stock	α (%)	Jumps	β_j t-stat		R^2 (J-R^2)			$RMSE$ (J-$RMSE$)		
			HAR-CJ	HAR-TCJ	HAR	HAR-CJ	HAR-TCJ	HAR	HAR-CJ	HAR-TCJ
aa	99.9	121	0.684	2.618	0.581	0.585	0.577	0.234	0.232	0.234
					(0.531)	(0.540)	(0.529)	(0.283)	(0.267)	(0.261)
	99.99	68	-0.556	1.818	0.581	0.585	0.578	0.234	0.232	0.234
					(0.436)	(0.463)	(0.460)	(0.279)	(0.246)	(0.244)
c	99.9	105	0.069	4.105	0.815	0.818	0.817	0.233	0.230	0.231
					(0.792)	(0.804)	(0.825)	(0.281)	(0.230)	(0.209)
	99.99	59	-0.202	4.183	0.815	0.819	0.818	0.233	0.230	0.230
					(0.810)	(0.828)	(0.845)	(0.304)	(0.229)	(0.212)
intc	99.9	78	3.101	4.908	0.804	0.804	0.800	0.214	0.215	0.216
					(0.823)	(0.829)	(0.828)	(0.215)	(0.199)	(0.206)
	99.99	43	1.866	3.958	0.804	0.804	0.802	0.214	0.214	0.215
					(0.790)	(0.807)	(0.813)	(0.244)	(0.218)	(0.225)
msft	99.9	92	0.197	2.130	0.796	0.796	0.795	0.227	0.227	0.227
					(0.738)	(0.728)	(0.741)	(0.286)	(0.262)	(0.246)
	99.99	48	0.456	1.859	0.796	0.795	0.794	0.227	0.227	0.227
					(0.605)	(0.593)	(0.620)	(0.335)	(0.313)	(0.294)
pfe	99.9	131	0.422	2.987	0.540	0.545	0.544	0.263	0.261	0.261
					(0.574)	(0.583)	(0.601)	(0.310)	(0.268)	(0.241)
	99.99	83	0.420	2.612	0.540	0.544	0.543	0.263	0.261	0.260
					(0.526)	(0.550)	(0.588)	(0.339)	(0.283)	(0.253)
xom	99.9	98	0.604	2.687	0.681	0.683	0.681	0.234	0.233	0.234
					(0.768)	(0.764)	(0.767)	(0.213)	(0.194)	(0.188)
	99.99	54	0.451	2.147	0.681	0.683	0.682	0.234	0.233	0.233
					(0.720)	(0.708)	(0.716)	(0.256)	(0.229)	(0.216)

Table 9: Reports number of jumps, t-stat on β_j, R^2, $RMSE$, $J - R^2$ and $J - RMSE$ for daily ($h = 1$) **logarithmic** version of HAR, HAR-CJ and HAR-TCJ volatility forecast regressions on six single stocks . The significant daily jump are computed using a critical value of $\alpha = 0.999$ and $\alpha = 0.9999$ as reported, with the C-Tz statistics.

6.3 Bond data and the no-trade bias

Finally, we use a sample of 30-years US Treasury Bond futures from January 1990 to October 2003 for a total of $3,231$ daily data points. All the relevant volatility and jump measures are computed again with five-minutes returns, corresponding to 84 returns per day.

The first thing we note on bond data is an unnaturally high number of jumps. At the 99.9% confidence level, the C-Tz statistics detects 570 jumps, corresponding to the 17.6 % of our sample. Visual inspection of time series data reveals that in most of these days there are not jumps at all, but many intervals with zero return instead.

The problem hinges from what we could call *the no-trade bias* of bipower variation. This can be explained as follows. Suppose that data are not recorded continuously but, more realistically, that they are recorded discretely. Denote by $\bar{\bar{\delta}}$ the minimum distance between two subsequent observations. By construction, if $\delta < \bar{\bar{\delta}}$, then $\mathsf{MPV}_\delta = 0$ identically! Clearly, also $\mathsf{TMPV}_\delta = 0$. This simple reasoning also explains why the presence of null returns caused by absence of trading (stale price) in that interval induces a downward bias in multipower variation measures. Note that realized volatility is immune from this bias instead. Moreover, this bias has a larger impact on the jump detecting statistics, pointing toward rejection of the null. For example, when considering the z statistics, this bias lowers both the BPV and TriPV measure, with the joint effect of increasing the numerator and decreasing the denominator, thus increasing the statistics considerably.

In our paper, this problem does not affect the S&P500 index, neither the stocks considered in our empirical analysis. However, it may affect US bond data, which are largely less liquid. Indeed, the percentage of zero 5-minutes return in bond data is very high, nearly 30%.

We accommodate this problem as follows: for bond data, we compute the C-Tz statistics using only returns different from zero. Clearly, this biases the test toward the null, meaning that the detected jumps are those which have a larger impact for the statistics. With this correction the number of significant jumps with $\alpha = 99.9\%$ reduced to 112 representing 3.4% of the sample.

Relevant estimation results for bond data when forecasting daily, weekly and monthly volatility are shown in Table 10 for $\alpha = 99.9\%$. Corresponding quantities for other values of α are shown in Figure 8. We find that the HAR-TCJ model outperforms the HAR-CJ model. This is true even if the impact of the jump on future volatility is generally insignificant, but nevertheless not negatively significant as for the HAR-CJ estimates. An explanation for this finding might be that jumps in the bond market are less "surprising" with respect to those in the stock markets, since most of them are related to scheduled macroeconomic announcements. Indeed, related studies, for example Bollerslev et al. (2000), find two intraday spikes at hours in which announcements are released. However, regarding forecasting, we confirm the results on S&P500 and individual stocks.

US Bond Regressions (C-Tz statistics)

HAR: $\mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d \mathsf{RV}_t + \beta_w \mathsf{RV}_{t-5:t} + \beta_m \mathsf{RV}_{t-22:t} + \varepsilon_t$

HAR-CJ: $\mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d \widehat{C}_t + \beta_w \widehat{C}_{t-5:t} + \beta_m \widehat{C}_{t-22:t} + \beta_j \widehat{J}_t + \varepsilon_t$

HAR-TCJ: $\mathsf{RV}_{t+1:t+h} = \beta_0 + \beta_d \widehat{TC}_t + \beta_w \widehat{TC}_{t-5:t} + \beta_m \widehat{TC}_{t-22:t} + \beta_j \widehat{TJ}_t + \varepsilon_t$

	RV_{t+1}			$\mathsf{RV}_{t+1}^{1/2}$			$log\,\mathsf{RV}_{t+1}$		
	HAR	HAR-CJ	HAR-TCJ	HAR	HAR-CJ	HAR-TCJ	HAR	HAR-CJ	HAR-TCJ
Daily forecasts									
β_j		-0.095	0.017		-0.038	0.029		-0.010	0.022
		(-2.050)	(0.711)		(-1.339)	(1.355)		(-0.692)	(1.813)
R^2	0.124	0.143	0.145	0.204	0.215	0.217	0.250	0.255	0.258
MSE	0.349	0.340	0.339	0.308	0.303	0.302	0.283	0.281	0.280
J-R^2	0.034	0.056	0.051	0.067	0.077	0.078	0.088	0.086	0.097
J-MSE	0.486	0.375	0.343	0.413	0.337	0.326	0.355	0.316	0.309
C-R^2	0.133	0.146	0.148	0.211	0.219	0.222	0.256	0.261	0.264
C-MSE	0.343	0.339	0.339	0.303	0.302	0.301	0.280	0.280	0.279
Weekly forecasts									
β_j		-0.071	0.012		-0.050	0.013		-0.026	0.007
		(-3.182)	(1.324)		(-3.363)	(1.301)		(-3.225)	(1.101)
R^2	0.295	0.336	0.343	0.415	0.439	0.445	0.472	0.487	0.492
MSE	0.206	0.197	0.196	0.174	0.170	0.169	0.163	0.161	0.160
J-R^2	0.126	0.146	0.156	0.261	0.277	0.289	0.334	0.340	0.353
J-MSE	0.162	0.159	0.158	0.176	0.175	0.177	0.197	0.197	0.199
C-R^2	0.312	0.356	0.365	0.426	0.451	0.458	0.480	0.494	0.500
C-MSE	0.208	0.199	0.197	0.174	0.170	0.169	0.162	0.160	0.159
Monthly forecasts									
β_j		-0.059	0.001		-0.049	-0.002		-0.026	-0.002
		(-3.999)	(0.206)		(-4.597)	(-0.279)		(-4.372)	(-0.537)
R^2	0.333	0.383	0.396	0.433	0.466	0.481	0.488	0.509	0.522
MSE	0.168	0.160	0.157	0.135	0.131	0.130	0.128	0.126	0.126
J-R^2	0.334	0.345	0.357	0.430	0.454	0.482	0.467	0.487	0.513
J-MSE	0.145	0.138	0.130	0.122	0.119	0.116	0.124	0.123	0.124
C-R^2	0.332	0.384	0.398	0.433	0.466	0.481	0.488	0.509	0.522
C-MSE	0.169	0.160	0.158	0.136	0.131	0.130	0.128	0.127	0.126

Table 10: OLS (partial) estimates for daily ($h = 1$), weekly ($h = 5$), monthly ($h = 22$) HAR, HAR-CJ and HAR-TCJ volatility forecast regressions for US Bond from January 1990 to December 2004 (3,736 observations). The significant daily jump are computed using a critical value of $\alpha = 99.9\%$ and the C-Tz statistics. Reported in parenthesis are the t-statistics based on Newey-West correction.

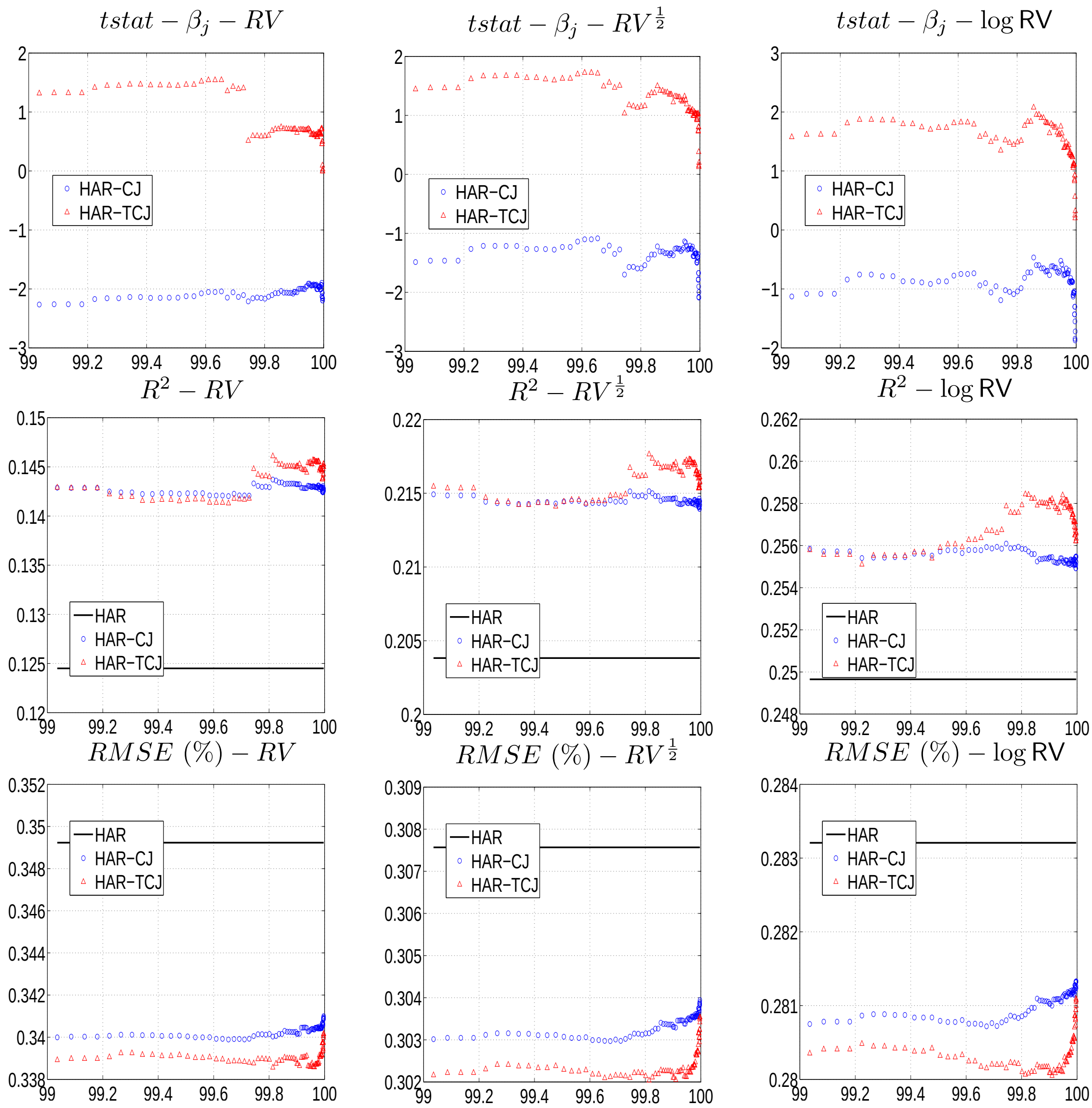

Figure 8: Reports the t statistics of the coefficient β_j, the R^2 and the $RMSE$ for the three models estimated on US Bond data, for both the HAR-CJ and HAR-TCJ versions, as a function of the confidence level used for detecting jumps with the C-Tz statistics.

Summarizing, our empirical findings further corroborate the theoretical and simulation results in the previous sections on the superior performance of the threshold method in separating and estimating the continuous and jump components of the price process. Moreover, they show that, once the two components are correctly measured and separated, the impact of past jumps on future realized volatility is positive and significant.

7 Conclusions

This paper shows that dividing volatility into jumps and continuous variation yields a substantial improvement in volatility forecasting, because of the positive impact of past jumps on future volatility. This important result has been obscured in the literature since, in finite samples, measures based on multipower variation are largely biased in the presence of jumps. We uncover this effect by modifying bipower variation with the help of threshold estimation techniques. We show that the newly defined estimator is robust to the presence of jumps and quite unelastic with respect to the choice of the threshold. Our empirical results, obtained on US stock index, single stocks and Treasury bond data, also show that jumps can be effectively detected using the newly proposed C-Tz statistics. The models we propose provide a superior forecasting ability, especially in days which follow the occurrence of a jump. The forecasting power on subsequent volatilities extends for a period of at least one month. Clearly, this finding can be of great importance for risk management and other financial applications involving volatility estimation.

These findings can also be important for the following reason. Recently, the financial econometrics literature focused on the separation of the quadratic variation in its discontinuous and continuous part. While ingenuous, sophisticated and fascinating theories have been contrived to this purpose, these had still little impact on empirical applications. Our study contributes to showing that the above mentioned separation theories (multipower variation, threshold estimation) can indeed be very useful in practical applications, since the two components seem to have different dynamics, with the continuous one being determined endogenously in the market by heterogeneous agents, and the discontinuous one being exogenous and basically unpredictable. Moreover, the correlation between jumps and future volatility can be helpful not only for practical applications, but also in the comprehension of the price formation mechanism. Some papers, like Eraker et al. (2003); Liu et al. (2003); Broadie et al. (2007); Todorov and Tauchen (2006) develop models which are consistent with our findings. Still, an economic theory explaining our findings may represent an interesting direction for future research.

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A Quarticity estimators and jump detection statistics

For estimation of the integrated quarticity, the literature focused on the following quantities:

$$\mathsf{QPV}_\delta(X)_t = \mu_1^{-4} \cdot \mathsf{MPV}_\delta(X)_t^{[1,1,1,1]} = \mu_1^{-4} \frac{1}{\delta} \sum_{j=4}^{[T/\delta]} |\Delta_{j-3}X| \cdot |\Delta_{j-2}X| \cdot |\Delta_{j-1}X| \cdot |\Delta_j X| \underset{\delta\to 0}{\longrightarrow} \int_t^{t+T} \sigma_s^4 ds \tag{A.1}$$

and

$$\mathsf{TriPV}_\delta(X)_t = \mu_{\frac{4}{3}}^{-3} \cdot \mathsf{MPV}_\delta(X)_t^{[\frac{4}{3},\frac{4}{3},\frac{4}{3}]} = \mu_{\frac{4}{3}}^{-3} \frac{1}{\delta} \sum_{j=3}^{[T/\delta]} |\Delta_{j-2}X|^{\frac{4}{3}} \cdot |\Delta_{j-1}X|^{\frac{4}{3}} \cdot |\Delta_j X|^{\frac{4}{3}} \underset{\delta\to 0}{\longrightarrow} \int_t^{t+T} \sigma_s^4 ds \tag{A.2}$$

where $\mu_{\frac{4}{3}} \simeq 0.8309$.

Barndorff-Nielsen and Shephard (2006) develop a theory, based on multipower variation, that allows to test for the presence of jumps in a time window $[0, T]$ with the desired level of significance. The following proposition defines the proper statistic to test for jumps.

Proposition A.1 (Barndorff-Nielsen and Shephard (2006)) *Assume that:*

1. $c_t dN_t = 0$ *in model (2.1).*
2. *The volatility process is bounded away from zero.*
3. *The joint process* (μ, σ) *is independent of the Brownian motion* W *driving the variable* X*.*

then, as $\delta \to 0$*, we have:*

$$G = \frac{\delta^{-\frac{1}{2}} \left(\mu_1^{-2}\,\mathsf{MPV}_\delta\,(X)_T^{[1,1]} - \mathsf{RV}_\delta\,(X)_T\right)}{\sqrt{\bar{\vartheta}\,\int_0^T \sigma_s^4\,ds}} \to^L N(0,1) \tag{A.3}$$

and

$$H = \frac{\delta^{-\frac{1}{2}} \left(\frac{\mu_1^{-2}\,\mathsf{MPV}_\delta(X)_T^{[1,1]}}{\mathsf{RV}_\delta(X)_T} - 1\right)}{\sqrt{\bar{\vartheta}\,\frac{\int_0^T \sigma_s^4\,ds}{\left(\int_0^T \sigma_s^2\,ds\right)^2}}} \to^L N(0,1)\,, \tag{A.4}$$

with $\bar{\vartheta} = \frac{\pi^2}{4} + \pi - 5$.

The same results with weaker assumptions have been reached by Barndorff-Nielsen et al. (2006). Proposition (A.1) is based on the fact that $p - \lim_{\delta\to 0} \mathsf{MPV}_\delta\,(X)_T^{[1,1]} = \mu_1^2 \int_0^T \sigma_s^2 ds$ and thus combining this result with equation (2.5) we have that:

$$\mu_1^{-2}\,\mathsf{MPV}_\delta\,(X)_T^{[1,1]} - \mathsf{RV}_\delta\,(X)_T \to^p -\sum_{j=1}^{N_T} c_j^2 \le 0, \tag{A.5}$$

$$\frac{\mu_1^{-2}\,\mathsf{MPV}_\delta\,(X)_T^{[1,1]}}{\mathsf{RV}_\delta\,(X)_T} - 1 \to^p -\frac{\sum_{j=1}^{N_T} c_j^2}{\int_0^T \sigma_s^2 ds + \sum_{j=1}^{N_T} c_j^2} \le 0. \tag{A.6}$$

A feasible jump test can be now constructed using multipower estimators of the integrated quarticity given by equations (A.1)-(A.2). In their study, ABD use the test:

$$z = \delta^{-\frac{1}{2}} \frac{\left(\mathsf{RV}_\delta\,(X)_T - \mathsf{BPV}_\delta\,(X)_T\right) \cdot \mathsf{RV}_\delta\,(X)_T^{-1}}{\sqrt{\bar{\vartheta}\,\max\left\{1, \frac{\mathrm{TriPV}_\delta(X)_T}{\left(\mathsf{BPV}_\delta(X)_T\right)^2}\right\}}}, \tag{A.7}$$

Monte Carlo studies of Huang and Tauchen (2005) showed that the statistics G and H in Proposition A.1 have a better power if implemented in their logarithmic forms. We then define the logarithmic jump test statistics as follows:

$$z_1 = \delta^{-\frac{1}{2}} \frac{\log \mathsf{RV}_\delta (X)_T - \log \mathsf{BPV}_\delta (X)_T}{\sqrt{\bar{\vartheta} \frac{\mathrm{QPV}_\delta(X)_T}{\left(\mathsf{BPV}_\delta(X)_T\right)^2}}}, \tag{A.8}$$

$$z_2 = \delta^{-\frac{1}{2}} \frac{\log \mathsf{RV}_\delta (X)_T - \log \mathsf{BPV}_\delta (X)_T}{\sqrt{\bar{\vartheta} \frac{\mathrm{TriPV}_\delta(X)_T}{\left(\mathsf{BPV}_\delta(X)_T\right)^2}}}. \tag{A.9}$$

In what follows, we will use the following special cases of threshold multipower variation, which are the counterparts of (A.1) and (A.2):

$$\mathsf{TQPV}_\delta(X)_t = \mu_1^{-4} \cdot \mathsf{TMPV}_\delta(X)_t^{[1,1,1,1]} \underset{\delta\to 0}{\longrightarrow} \int_t^{t+T} \sigma_s^4 ds, \tag{A.10}$$

$$\mathsf{TTriPV}_\delta(X)_t = \mu_{\frac{4}{3}}^{-3} \cdot \mathsf{TMPV}_\delta(X)_t^{[\frac{4}{3},\frac{4}{3},\frac{4}{3}]} \underset{\delta\to 0}{\longrightarrow} \int_t^{t+T} \sigma_s^4 ds. \tag{A.11}$$

The theory of threshold multipower variation allows for additional tests, which are the natural counterparts of the statistics (A.8),(A.9). We then introduce the corresponding threshold z statistics, defined as:

$$\mathsf{Tz} = \delta^{-\frac{1}{2}} \frac{\left(\mathsf{RV}_\delta (X)_T - \mathsf{TBPV}_\delta (X)_T\right) \cdot \mathsf{RV}_\delta (X)_T^{-1}}{\sqrt{\bar{\vartheta} \max\left\{1, \frac{\mathrm{TTriPV}_\delta(X)_T}{\left(\mathsf{TBPV}_\delta(X)_T\right)^2}\right\}}}, \tag{A.12}$$

$$\mathsf{Tz}_1 = \delta^{-\frac{1}{2}} \frac{\log \mathsf{RV}_\delta (X)_T - \log \mathsf{TBPV}_\delta (X)_T}{\sqrt{\bar{\vartheta} \frac{\mathrm{TQPV}_\delta(X)_T}{\left(\mathsf{TBPV}_\delta(X)_T\right)^2}}}. \tag{A.13}$$

$$\mathsf{Tz}_2 = \delta^{-\frac{1}{2}} \frac{\log \mathsf{RV}_\delta (X)_T - \log \mathsf{TBPV}_\delta (X)_T}{\sqrt{\bar{\vartheta} \frac{\mathrm{TTriPV}_\delta(X)_T}{\left(\mathsf{TBPV}_\delta(X)_T\right)^2}}}. \tag{A.14}$$

and their corrected versions, which performs well under the null of no jumps, as discussed in Section 2.4:

$$\mathsf{C\text{-}Tz}_1 = \delta^{-\frac{1}{2}} \frac{\log \mathsf{RV}_\delta (X)_T - \log \mathsf{C\text{-}TBPV}_\delta (X)_T}{\sqrt{\bar{\vartheta} \frac{\mathsf{C\text{-}TQPV}_\delta(X)_T}{\left(\mathsf{C\text{-}TBPV}_\delta(X)_T\right)^2}}}, \tag{A.15}$$

$$\mathsf{C\text{-}Tz}_2 = \delta^{-\frac{1}{2}} \frac{\log \mathsf{RV}_\delta (X)_T - \log \mathsf{C\text{-}TBPV}_\delta (X)_T}{\sqrt{\bar{\vartheta} \frac{\mathsf{C\text{-}TTriPV}_\delta(X)_T}{\left(\mathsf{C\text{-}TBPV}_\delta(X)_T\right)^2}}}. \tag{A.16}$$