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Nonparametric Quantile Regression with Heavy-Tailed and Strongly Dependent Errors

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Abstract

We consider nonparametric estimation of the conditional qth quantile for stationary time series. We deal with stationary time series with strong time dependence and heavy tails under the setting of random design. We estimate the conditional qth quantile by local linear regression and investigate the asymptotic properties. It is shown that the asymptotic properties are affected by both the time dependence and the tail index of the errors. The results of a small simulation study are also given.

Key words: conditional quantile, random design, check function, local linear regression, stable distribution, linear process, long-range dependence, martingale central limit theorem

1. Introduction

Let $\{(X_i, Y_i)\}$ be a bivariate stationary process generated by

$$Y_i = u(X_i) + V_i, \quad i = 1, 2, \dots, \tag{1}$$

where $V_i = V(X_i, Z_i)$, $X_i = J(\dots, \epsilon_{i-1}, \epsilon_i)$, $Z_i = \sum_{j=0}^{\infty} c_j \zeta_{i-j}$, and $\{\epsilon_i\}$ and $\{\zeta_i\}$ are mutually independent i.i.d. processes. Then we estimate the qth conditional quantile of Y_i given $X_i = x_0$ from n observations by appealing to local linear regression and investigate the asymptotic properties of the estimator.

We adopt the DGP and the dependence measure of Wu et al. (2010), which allows us to consider nonlinearity and long-range dependence (LRD).

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We can also deal with $\{X_i\}$ that does not satisfy α-mixing conditions sufficient for the central limit theorem. See Wu et al.(2010) for the details. Assuming that $\{Z_i\}$ is a heavy-tailed linear process and c_j does not decay so fast, we examine how the heavy tail and the time dependence through $\{c_j\}$ affect the asymptotic properties of the local linear estimator in the setting of (1). We need the assumption of linear process as in (1) to derive the asymptotic distribution of the estimator.

We state a few assumptions on $u(x)$ and $V(x,z)$ here. Let $u(x)$ be twice continuously differentiable in a neighborhood of x_0. We denote the qth quantile of Z_1 by m_q and assume that $V(x,z)$ is monotone increasing in z and $V(x,m_q)=0$ for any x. Then $u(x_0)$ is the conditional qth quantile given $X_i=x_0$. An example of $V(x,z)$ is $\sigma(x)(z-m_q)$. Some more technical assumptions on $V(x,z)$ will be given in section 2.

There have been a lot of studies on quantile regression for linear models since Koenker and Basset (1978). It is because quantile regression gives us more information about data than mean regression and is robust to outliers. Pollard (1991) devised a simple proof of the asymptotic normality of regression coefficient estimators. See Koenker (2005) for recent developments of quantile regression.

We often employ nonparametric regression when we have no parametric regression function or when we want to check the parametric regression function. Chaudhuri (1991) considered nonparametric estimation of conditional quantiles for i.i.d observations and Fan et al. (1994) applied the method of Pollard (1991) to nonparametric robust estimation including nonparametric estimation of conditional quantiles. We examine the estimator of Chaudhuri (1991) in our setting by exploiting the method of Pollard (1991). See Fan and Gijbels (1996) for nonparametric regression and local linear estimators.

Many authors have considered cases of weakly dependent observations and studied the asymptotic properties of the nonparametric quantile estimators since Chaudhuri (1991). For example, Truong and Stone (1992) considered local medians for α-mixing processes. Honda (2000a) and Hall et al.(2002) examined the asymptotic properties of the estimator of Chaudhuri (1991). Hall et al.(2002) also employed the method of Pollard (1991) for α-mixing processes. Zhao and Wu (2006) considered another setting from α-mixing processes. The above authors considered nonparametric quantile estimation under random design. Zhou (2010) is a recent paper for nonparametric quantile estimation under fixed design. See Fan and Yao (2003) for nonparametric regression for time series.

Some authors investigated robust or nonparametric estimation of regression functions for LRD time series with finite variance after the developments of theoretical results on time series with LRD, especially, the results on linear processes by Ho and Hsing (1996,1997). Giraitis et al. (1996) deals with robust linear regression under LRD. See Robinson (1997), Hidalgo (1997), Csörgő and Mielniczuk (2000), Mielniczuk and Wu (2004), and Guo and Koul (2007) for nonparametric estimation of conditional mean functions. Wu and Mielniczuk (2002) fully examined the asymptotic properties of kernel density estimators. Wu et al. (2010) also deals with kernel density estimation and nonparametric regression and the results are useful to the present paper. Honda (2000b) and Honda (2010) considered nonparametric estimation of conditional quantiles when $\{X_i\}$ and $\{Z_i\}$ are LRD linear processes with finite variance in (1). It is now known that the asymptotic distributions of nonparametric estimators drastically change depending on the strength of dependence and the bandwidths in the cases of density estimation and nonparametric regression under random design. The time dependence of covariates has almost no effect on the asymptoitics except for technical conditions in the setting similar to (1). See Beran (1994), Robinson (2003), and Doukhan et al.(2003) for surveys on time series with LRD.

Following Ho and Hsing (1996,1997), Hsing (1999), Koul and Surgailis (2001), Surgailis (2002), Pipiras and Taqqu (2003), and Honda (2009b) studied the limiting distributions of partial sums of bounded functionals of LRD linear processes with infinite variance. We state Assumptions Z1-2 on $\{Z_i\}$ to describe their results. Let $a_n \sim a_n'$ mean $a_n/a_n' \to 1$ as $n \to \infty$.

Assumption Z1: $c_j \sim c_z j^{-\beta}$ and $c_0 = 1$.

Assumption Z2: Write $G_0(z)$ for the distribution function of ζ_1. Then there exists $0 < \alpha < 2$ s.t.

$$\lim_{z\to-\infty} |z|^\alpha G_0(z) = c_- \quad \text{and} \quad \lim_{z\to\infty} |z|^\alpha (1 - G_0(z)) = c_+,$$

where $c_- + c_+ > 0$. In addition, $\mathrm{E}\{\zeta_1\} = 0$ when $\alpha > 1$.

Hereafter we assume that Assumptions Z1-2 hold. Then there are three Cases in the literature and we summarize the cases and the references in Table 1 below. Some authors say that the linear process has LRD in Cases 1-2. Note that ζ_1 belongs to the domain of attraction of the α-stable distribution $S_\alpha(\sigma, \eta, \mu)$, whose characteristic function is given by

$$\begin{cases} \exp\{-\sigma^\alpha|\theta|^\alpha(1 - \mathrm{i}\eta\,\mathrm{sign}(\theta)\tan(\pi\alpha/2)) + \mathrm{i}\mu\theta\} & \text{for } \alpha \neq 1, \\ \exp\{-\sigma|\theta|(1 + \frac{2}{\pi}\mathrm{i}\eta\,\mathrm{sign}(\theta)\log|\theta|) + \mathrm{i}\mu\theta\} & \text{for } \alpha = 1, \end{cases}$$

Table 1: 3 Cases for α and β

Case 1	**Case 2**	**Case 3**
$1 < \alpha < 2$	$0 < \alpha < 2$	$0 < \alpha < 2$
$1/\alpha < \beta < 1$	$1 < \beta < 2/\alpha$	$2/\alpha < \beta$
Koul and Surgailis (2001)	Surgailis (2002)	Hsing (1999)
	Honda (2009b)	Pipiras and Taqqu (2003)

where $0 < \sigma$, $-1 \leq \eta \leq 1$, $-\infty < \mu < \infty$, and i stands for the imaginary unit. See Samorodnitsky and Taqqu (1994) for more details about stable distributions.

In Case 3, we have

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (H(Z_i) - \mathrm{E}\{H(Z_i)\}) \stackrel{d}{\to} \mathrm{N}(0, \sigma^2),$$

where $\stackrel{d}{\to}$ denotes convergence in distribution and $H(z)$ is a bounded function. In Cases 1 and 2, the limiting distribution is an α- and $\alpha\beta$-stable distribution with $n^{-1+\beta-1/\alpha}$ and $n^{-1/(\alpha\beta)}$ as the normalization constant, respectively.

Some authors have considered robust parametric or nonparametric estimation under dependent errors with infinite variance, i.e. in Case1, Case2 with $\alpha > 1$, and Case 3. Peng and Yao (2004) and Chan and Zhang (2009) considered robust nonparametric regression under fixed design. Honda (2009a) considered kernel density estimation by following Wu and Mielniczuk (2002) and found that the asymptotic distributions depend on α and β in Assumptions Z1-2. Koul and Surgalis (2001) and Zhou and Wu (2010) deals with linear regression in Case 1.

In this paper, we consider nonparametric estimation of the conditional qth quantile in (1) in Cases 1-3 by following Honda (2010). We can also say that this paper is a random-design version of Peng and Yao (2004) and Chan and Zhang (2009). Then we find that α and β affect the asymptotics in Cases 1-2 and that we have the same asymptotics as for i.i.d. observations in Case 3. We also conclude that the time dependence of $\{X_i\}$ has almost no effect on the asymptotics in Cases 1 and 3. The Case 2 is the most challenging and we have not resolved the effect of the LRD of $\{X_i\}$ completely. See Theorem

2 below for more details. We conjecture that the strong LRD of $\{X_i\}$ affects the asymptotics of the estimator. However, this is a topic of future research.

This paper is organized as follows. In Section 2, we describe assumptions, define the local linear estimator, and present the asymptotic properties in Theorems 1-3. We carried out a small simulation study and the results are reported in Section 3. We state Propositions 1-5 and prove Theorems 1-3 in Section 4. All the technical details are confined to Sections 5-6.

Finally in this section, we introduce some notation. We write $|w|$ and A^T for the Euclidean norm of a vector w and the transpose of a matrix A. We denote the L_p norm of a random variable W by $\|W\|_p$ and p is omitted when $p = 2$. Let $\stackrel{p}{\to}$ denote convergence in probability and we omit a.s. (almost surely) when it is clear from the context.

We write $a \wedge b$ and $a \vee b$ for $\min\{a, b\}$ and $\max\{a, b\}$, respectively. Let $\mathbb{R}$ and $\mathbb{Z}$ denote the set of real numbers and integers, respectively. Throughout this paper, C and δ are positive generic constants and the values vary from place to place. The range of integration is also omitted when it is $\mathbb{R}$.

2. Local linear estimator and asymptotic properties

We state assumptions, define the local linear estimator, and present the asymptotic properties of the estimator in Theorems 1-3.

First we state Assumption V on $V(x, z)$. Recall that m_q is the qth quantile of Z_1

Assumption V: $V(x, z)$ is monotone increasing in z and $V(x, m_q) = 0$ for any x. Besides, $V(x, z)$ is continuously differentiable in a neighborhood of (x_0, m_q) and $\partial V(x_0, m_q)/\partial z > 0$.

We need a kernel function $K(\xi)$ and a bandwidth h to define the local linear estimator.

Assumption K: The kernel function $K(\xi)$ is a symmetric and bounded density function with compact support $[-C_K, C_K]$. We write κ_j and ν_j for $\int \xi^j K(\xi)d\xi$ and $\int \xi^j K^2(\xi)d\xi$, respectively.

Assumption H: $h = c_h n^{-1/5}$ for some positive c_h.

We impose Assumption H for simplicity of presentation. However, other choices of h do not improve the rate of convergence of the estimator. There is no theoretical difficulty in dealing with the case where $X_i \in \mathbb{R}^d$. Then we should take $h = c_h n^{-1/(d+4)}$.

Now we introduce the check function $\rho_q(u)$ and the derivative $\rho_q'(u)$ in (2) to define the local linear estimator of $u(x_0)$.

$$\rho_q(u) = u(q - I(u < 0)) \quad \text{and} \quad \rho_q'(u) = q - I(u < 0). \tag{2}$$

Then we estimate $(u(x_0), hu'(x_0))^T$ by

$$\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2)^T = \mathrm{argmin}_{\beta \in \mathbb{R}^2} \sum_{i=1}^n K_i \rho_q(Y_i - \eta_i^T \beta),$$

where $K_i = K((X_i - x_0)/h)$ and $\eta_i = (1, (X_i - x_0)/h)^T$.

We normalize $\hat{\beta} - (u(x_0), hu'(x_0))^T$ by τ_n and define $\hat{\theta}$ by

$$\hat{\theta} = \tau_n(\hat{\beta}_1 - u(x_0), \hat{\beta}_2 - hu'(x_0))^T. \tag{3}$$

We specify τ_n later in this section. It is easy to see that $\hat{\theta}$ is also defined by

$$\hat{\theta} = \mathrm{argmin}_{\theta \in \mathbb{R}^2} \sum_{i=1}^n K_i \rho_q(V_i^* - \tau_n^{-1} \eta_i^T \theta), \tag{4}$$

where

$$V_i^* = V(X_i, Z_i) + \frac{1}{2}\Big(\frac{X_i - x_0}{h}\Big)^2 u''(\bar{X}_i)$$

and $\bar{X}_i$ is between x_0 and X_i.

Before stating assumptions on $\{X_i\}$ and $\{Z_i\}$, we define σ-fields $\mathcal{F}_i$, $\mathcal{G}_i$, and $\mathcal{S}_i$ by

$$\mathcal{F}_i = \sigma(\dots, \epsilon_{i-1}, \epsilon_i), \quad \mathcal{G}_i = \sigma(\dots, \zeta_{i-1}, \zeta_i), \ \mathcal{S}_i = \sigma(\dots, \epsilon_{i-1}, \zeta_{i-1}, \epsilon_i, \zeta_i).$$

We adopt the setup and the notation of Wu et al.(2010), especially that of subsection 2.1, for $\{X_i\}$ and Assumption X1 below is necessary to define the dependence measure.

Set

$$F_l(x|\mathcal{F}_i) = \mathrm{P}(X_{i+l} \le x|\mathcal{F}_i). \tag{5}$$

Assumption X1: With probability 1, $F_1(x|\mathcal{F}_0)$ is differentiable on $\mathbb{R}$ and the derivative $f_1(x|\mathcal{F}_0)$ satisfies $\sup_{\mathbb{R}} f_1(x|\mathcal{F}_0) \le C$ and $\lim_{x \to x_0} \mathrm{E}\{|f_1(x|\mathcal{F}_0) - f_1(x_0|\mathcal{F}_0)|\} = 0$.

We write $f(x)$ for the density function of X_1 and assume that $f(x_0) > 0$ throughout the paper. Here notice that $f(x) = \mathrm{E}\{f_1(x|\mathcal{F}_0)\}$.

Another σ-field $\mathcal{F}_i^*$ below is used to define the dependence measure of $\{X_i\}$ as in Wu et al.(2010).

$$\mathcal{F}_i^* = \begin{cases} \sigma(\ldots, \epsilon_{-1}, \epsilon_0^*, \epsilon_1, \ldots, \epsilon_i) & \text{for } i \geq 0, \\ \mathcal{F}_i & \text{for } i < 0, \end{cases}$$

where ϵ_0^* is an independent copy of ϵ_0. Define the dependence measure $\theta_{j,p}(x)$ by

$$\theta_{j,p}(x) = \| f_{1+j}(x|\mathcal{F}_0) - f_{1+j}(x|\mathcal{F}_0^*)\|_p.$$

for $p > 1$ and $j \geq 0$. When $j < 0$, set $\theta_{j,p}(x) = 0$. Then we have

$$\|\mathrm{E}\{f_1(x|\mathcal{F}_i)|\mathcal{F}_{i-j}\} - \mathrm{E}\{f_1(x|\mathcal{F}_i)|\mathcal{F}_{i-j-1}\}\|_p \leq \theta_{j,p}(x). \tag{6}$$

We also define p', $\theta_p(j)$, and Θ_p by $p' = 2 \wedge p$,

$$\theta_p(j) = \sup_{x \in \mathbb{R}} \theta_{j,p}(x), \quad \text{and} \quad \Theta_p(n) = \sum_{i \in \mathbb{Z}} (\sum_{j=1-i}^{n-i} \theta_p(j))^{p'}. \tag{7}$$

We find in Subsection 4.1 of Wu et al.(2010) that $\theta_p(j) \leq C|b_j|$ for $1 < p \leq 2$ when $\mathrm{E}\{|\epsilon_i|^2\} < \infty$ and X_i is given by

$$X_i = \sum_{j=0}^{\infty} b_j \epsilon_{i-j}. \tag{8}$$

Assumption X2: $(\Theta_p(n))^{1/p'}/n \to 0$ for some $1 < p$.

Assumption X2 will be employed to deal with $\sum_{i=1}^n (f_1(x_0 + \xi h|\mathcal{F}_{i-1}) - f(x_0 + \xi h))$. In fact, Lemma 3 of Wu et al.(2010) implies that

$$\sup_{x \in \mathbb{R}} \| \sum_{i=1}^n (f_1(x|\mathcal{F}_{i-1}) - f(x)) \|_p \leq C(\Theta_p(n))^{1/p'} \tag{9}$$

and that almost every linear process with finite variance satisfies Assumption X2. We assume that Assumptions X1-2 hold throughout the paper.

Assumptions X3-5 below will be used to derive the asymptotic distribution when the effects of α and β appear in the asymptotics.

Assumption X3: $\sum_{j=1}^{\infty} \theta_p(j) < \infty$.

Assumption X3 means that $\{X_i\}$ has short-range dependence (SRD). It is easy to see that Assumption X3 implies Assumption X2. We take $p = \alpha$ and $\alpha\beta < p \leq 2$ in Cases 1 and 2, respectively.

Hereafter we write $A_\xi(i)$ for $f_1(x_0 + \xi h|\mathcal{F}_{i-1})$ for notational convenience. Recall that $\mathrm{E}\{A_\xi(i)\} = f(x_0 + \xi h)$. Assumption X4 below holds under (8) with $b_j \sim c_X j^{-(1+\delta_1)/2}$ and $\mathrm{E}\{|\epsilon_1|^{2+\delta_2}\} < \infty$ for some positive δ_1 and δ_2. Thus it is just a mild assumption and will be used in Case 1.

Assumption X4: There exists a positive γ_x s.t.

$$|\mathrm{Cov}(A_\xi(i), A_\xi(j))| \leq C|i-j|^{-\gamma_x} \quad \text{for } i \neq j.$$

Assumption X5: There exist r_x and δ_x s.t. $\alpha\beta < r_x$, $\delta_x > 0$, and $\theta_{r_x}(j) \leq Cj^{-\delta_x - 1/(\alpha\beta)}$.

Assumption X5 will be used in Case 2. The assumption is rather restrictive because it depends on $\alpha\beta$. However, it seems very difficult to derive the asymptotic distribution without this kind of assumption when we see the effects of α and β. See a comment on this difficulty around (11) and (12) below.

We introduce some more notation to state another assumption on $\{Z_i\}$. We define $Z_{i,j}$ and $\tilde{Z}_{i,j}$ by

$$Z_{i,j} = \sum_{l=0}^{j} c_l \zeta_{i-l} \quad \text{and} \quad \tilde{Z}_{i,j} = Z_i - Z_{i,j} = \sum_{l=j+1}^{\infty} c_l \zeta_{i-l}$$

and let $G_j(z)$ denote the distribution function of $Z_{1,j}$. Then $G_\infty(z)$ is that of Z_1.

Assumption Z3: There exists a positive γ_z s.t. for any j,

$$|G_j''(z)| \leq C(1+|z|)^{-(1+\gamma_z)} \quad \text{and} \quad |G_j''(z_1) - G_j''(z_2)| \leq \frac{C|z_1 - z_2|}{(1+|z_1|)^{(1+\gamma_z)}} \quad (10)$$

for $|z_1 - z_2| \leq 1$. In addition, $g_\infty(m_q) > 0$.

Assumption Z3 is a technical one and Lemma 4.2 of Koul and Surgailis (2001) implies that Assumption Z3 can be relaxed for $\alpha > 1$. When ζ_1 has a stable distribution, Assumption Z3 follows from the argument based on integration by parts in Hsing (1999).

We divide Case 1 into Cases 1A and 1B and Case 2 into Cases 2A-C, respectively to present Theorems 1-3. We also specify the normalization constant τ_n for each case here.

Case 1: $1 < \alpha < 2$, $1 < \alpha\beta < 2$, and $\beta < 1$

Case 1A: $1/\alpha - \beta < -2/5$ and $\tau_n = \sqrt{nh}$

Case 1B: $1/\alpha - \beta > -2/5$ and $\tau_n = n^{\beta-1/\alpha}$. In addition, Assumption X3 with $p = \alpha$ or X4 holds.

Case 2: $0 < \alpha < 2$, $1 < \alpha\beta < 2$, and $\beta > 1$

Case 2A: $1/(\alpha\beta) < 3/5$ and $\tau_n = \sqrt{nh}$.

Case 2B: $1/(\alpha\beta) > 3/5$ and $\tau_n = n^\nu$, where $\nu < 1 - 1/(\alpha\beta)$.

Case 2C: $1/(\alpha\beta) > 3/5$ and $\tau_n = n^{1-1/(\alpha\beta)}$. In addition, Assumption X3 with $\alpha\beta < p$ or X5 holds.

Case 3: $\alpha\beta > 2$ and $\tau_n = \sqrt{nh}$.

In Cases 1A, 2A, and 3, we have the same asymptotic distribution as for i.i.d. observations. On the other hand, we see the effects of α and β in Cases 1B, 2B, and 2C and have worse convergence rates. We have to impose additional assumptions on $\{X_i\}$ to investigate the asymptotic distribution of the nonparametric quantile estimator in those cases. Especially in Case 2, we have to show

$$n^{-1/(\alpha\beta)} \sum_{i=1}^{n} (A_\xi(i) - \mathrm{E}\{A_\xi(i)\}) B_1(\tilde{Z}_{i,0}) = o_p(1) \tag{11}$$

or to deal with

$$n^{-1/(\alpha\beta)} \sum_{i=1}^{n} \sum_{j=1}^{\infty} A_\xi(i+j)(B_j(c_j\zeta_i) - \mathrm{E}\{B_j(c_j\zeta_i)\}), \tag{12}$$

where $B_j(z)$ is specified later in Proposition 2. We will prove (11) and derive the asymptotic distribution in Case 2C. When (11) does not seem to hold, we have to deal with (12). However, $A_\xi(i+j)$ in (12), not $A_\xi(i)$, will extremely complicate the theoretical treatment and we do not pursue the problem in this paper.

Theorems 1-3 below deals with Cases 1-3, respectively. We denote the density of $V(x_0, Z_1)$ by $f_V(0|x_0)$, which is written as

$$f_V(0|x_0) = g_\infty(m_q)\Big(\frac{\partial V}{\partial z}(x_0, m_q)\Big)^{-1}.$$

Theorem 1. *Suppose that Assumptions V, K, H, Z1-3, and X1-2 hold in Case 1. In Case 1B, Assumption X3 with $p = \alpha$ or X4 is also assumed. Then we have as $n \to \infty$,*

Case 1A:

$$\hat{\theta} \xrightarrow{d} N\left(\left(\begin{array}{c} \frac{c_h^{5/2}u''(x_0)\kappa_2}{2} \\ 0 \end{array}\right), \frac{q(1-q)}{f_V^2(0|x_0)f(x_0)}\left(\begin{array}{cc} \nu_0 & 0 \\ 0 & \kappa_2^{-2}\nu_2 \end{array}\right)\right),$$

Case 1B:

$$\begin{aligned} \hat{\theta} &= -\frac{1}{f_V(0|x_0)}\left(\begin{array}{c} 1 \\ 0 \end{array}\right)\int \rho_q'(V(x_0,z))g_\infty'(z)dz\frac{\tau_n}{n}\sum_{i=1}^n \tilde{Z}_{i,0} + o_p(1) \\ &\xrightarrow{d} -\frac{1}{f_V(0|x_0)}\left(\begin{array}{c} 1 \\ 0 \end{array}\right)\int \rho_q'(V(x_0,z))g_\infty'(z)dzc_dL, \end{aligned}$$

where $\int \rho_q'(V(x_0,z))g_\infty'(z)dz = -g_\infty(m_q)$, $L \sim S_\alpha(1,(c_+ - c_-)/(c_+ + c_-),0)$, *and*

$$c_d = c_z\Big((c_+ + c_-)\frac{\Gamma(2-\alpha)\cos(\alpha\pi/2)}{1-\alpha}\int_{-\infty}^1 \Big\{\int_0^1 (t-s)_+^{-\beta}dt\Big\}ds\Big)^{1/\alpha}.$$

Theorem 2. *Suppose that Assumptions V, K, H, Z1-3, and X1-2 hold in Case 2. In Case 2C, Assumption X3 with* $\alpha\beta < p$ *or X5 is also assumed. Then we have as* $n \to \infty$*,*
Case 2A: *we have the same result as in Case 1A,*
Case 2B: $\hat{\theta} = o_p(1)$*,*
Case 2C:

$$\hat{\theta} \xrightarrow{d} \frac{\sigma_{\alpha\beta}}{f_V(0|x_0)}\left(\begin{array}{c} 1 \\ 0 \end{array}\right)(c_+^{1/(\alpha\beta)}C_q^+L^+ + c_-^{1/(\alpha\beta)}C_q^-L^-),$$

where $L^+ \sim S_{\alpha\beta}(1,1,0)$*,* $L^- \sim S_{\alpha\beta}(1,1,0)$*,* L^+ *and* L^- *are mutually independent,*

$$\begin{aligned} \sigma_{\alpha\beta} &= \Big\{\frac{c_z^\alpha\Gamma(2-\alpha\beta)|\cos(\pi\alpha\beta/2)|}{(\alpha\beta-1)\beta^{\alpha\beta}}\Big\}^{1/(\alpha\beta)}, \\ C_q^\pm &= \int_0^\infty (q - G_\infty(m_q \mp v)v^{-(1+1/\beta)}dv. \end{aligned}$$

In Case 2B, we have only proved that $\hat{\beta} - (u(x_0), hu'(x_0))^T = o_p(n^{-\nu})$ for any $\nu < 1 - 1/(\alpha\beta)$.

Theorem 3. *Suppose that Assumptions V, K, H, Z1-3, and X1-2 hold in Case 3. Then we have the same result as in Case 1A.*

Theorems 1-2 shows that the asymptotic properties may be badly affected by α and β in Cases 1B, 2B, and 2C. Generally speaking, the convergence rates of mean regression are worse than those of quantile regression when $\alpha < 2$. However, the convergence rate is the same as that of nonparametric mean regression in Case 1B. In Case 2, the rates are improved and better than $n^{-1+1/\alpha}$.

In Section 3, we report the results of our simulation study to show how α and β affect the properties of the local linear estimator.

In Cases 1A, 2A, and 3, our choice of h in Assumption H gives the optimal rate of convergence to the local linear estimator. In Cases 1B and 2C, the rate of convergence is independent of h and any other choices of h does not improve the rate. Therefore we recommend that we should choose the bandwidth as if we had i.i.d. observations.

The asymptotic distribution depends on α and β in a complicated way in Cases 1B and 2C. It might be very difficult to estimate the parameters and statistical inference is a topic of future research.

3. Simulation study

We carried out a small simulation study by using R. In the simulation study, $\epsilon_i \sim \mathrm{N}(0,1)$, $\eta_i \sim S_\alpha(1,0,0)$,

$$Y_i = 2(X_i^2 + X_i^4) + Z_i,\ X_i = \sum_{j=0}^{999} \frac{c_x}{(1+j)^\gamma}\epsilon_{i-j},\ Z_i = \sum_{j=0}^{999} \frac{c_z}{(1+j)^\beta}\eta_{i-j},$$

where c_x and c_z are chosen so that $X_i \sim \mathrm{N}(0,1)$ and $Z_i \sim S_\alpha(1,0,0)$.

We took $\gamma = 0.75$, 1.25, $x_0 = 0.0$, 0.6, and $h = 0.2$, 0.4. We examined 20 pairs of (α,β), $\alpha = 1.1, 1.2, 1.3, 1.4, 1.5$ and $\beta = 0.9, 1.3, 1.7, \infty$. The sample size is 400 and the results are based on 10,000 repetitions.

We estimate $u(x_0)$ by employing the rq function of the quantreg package(Koenker (2009)) with the Epanechnikov kernel and use the rstable function of the fBasics package(Wuertz et al.(2009)) to generate $S_\alpha(1,0,0)$ random numbers. When there are less than four observations available to estimate $u(x_0)$, just the sample median is used here. However, there are less than 10 of the repetitions for each entry of Tables 2-6 below and there will be almost no influence on the results.

Tables 2-6 are for the cases of $\alpha = 1.1, 1.2, 1.3, 1.4, 1.5$, respectively in the case of $\gamma = 0.75$. Tables 7-11 are for the same pairs with $\gamma = 1.25$. Note that all of $(*, 0.9)$ belong to Case 1B. Pairs $(1.1, 1.3)$ and $(1.2, 1.3)$ belong to Cases 2B and 2C in the cases of $\gamma = 0.75$ and $\gamma = 1.25$, respectively. The other pairs have the same asymptotic distribution as for i.i.d. observations. In the tables, every entry is estimated by the sample mean. "mean" is the mean of $\hat{\beta}_1$ and "bias" is the mean minus the true value. "mse" is the mean squared error and N/A means that the MSE does exist from a theoretical point of view. Actually, we had unstable and extremely large values. Values with $*$ in the tables were unstable and the true values may not exist. "madev" stands for the mean absolute deviation, $\mathrm{E}\{|\hat{\beta}_1 - u(x_0)|\}$.

Tables 2-11 are around here.

We have the following observations from Tables 2-11.

1. In the cases of $\beta = 0.9$, the values of madv are very large for small α. This implies that the effects of small β and small α are very serious and that nonparametric estimation may be very difficult.
2. In the cases of $\beta = 1.3$, the values of mse are large for $\alpha = 1.3 - 1.5$. We should have the same asymptotic distribution as for i.i.d. observations in those cases. The values of madv are still larger than those for $\beta = \infty$.
3. In the cases of $\beta = 1.7$, the effects of small α on mse are serious up to $\alpha = 1.3$ and the madv values are also severely affected up to $\alpha = 1.2$.
4. There are not large differences in the mean absolute deviation between $\gamma = 0.75$ and $\gamma = 1.25$. But there may be a difference in the MSE in $(1.2, 1.7)$.
5. Larger bandwidths yield better results for the MSE. But there is almost no difference in the mean absolute deviation between $h = 0.2$ and $h = 0.4$.

The effects of α and β are serious and there seem to be considerable differences between the asymptotics and the finite sample properties.

4. Proofs of Theorems 1-3

We verify Theorems 1-3 in a similar way to Theorem 1 of Honda (2010). Honda (2010) deals with linear process with finite variance. First we state Propositions 1-5, which are essential tools to the proofs. Propositions 1-3 deal with the stochastic term of the estimator and they correspond to

Lemma 1 of Honda (2010). Propositions 4 and 5 correspond to Lemmas 2 and 3, respectively and deal with all the cases at the same time. Proposition 5 is related to the bias term.

Proposition 1. *Suppose that the same assumptions hold as in Theorem 1. Then we have as $n \to \infty$,*
Case 1A:

$$\frac{\tau_n}{nh}\sum_{i=1}^{n} K_i \eta_i \rho_q'(V_i) \xrightarrow{d} N\left(\left(\begin{array}{c} 0 \\ 0 \end{array}\right), q(1-q)f(x_0)\left(\begin{array}{cc} \nu_0 & 0 \\ 0 & \nu_2 \end{array}\right)\right),$$

Case 1B:

$$\begin{aligned} & \frac{\tau_n}{nh}\sum_{i=1}^{n} K_i \eta_i \rho_q'(V_i) \\ = \quad & -f(x_0)\left(\begin{array}{c} 1 \\ 0 \end{array}\right)\int \rho_q'(V(x_0,z))g_\infty'(z)dz \cdot \frac{\tau_n}{n}\sum_{i=1}^{n}\tilde{Z}_{i,0} + o_p(1) \\ \xrightarrow{d} \quad & -f(x_0)\left(\begin{array}{c} 1 \\ 0 \end{array}\right)\int \rho_q'(V(x_0,z))g_\infty'(z)dz \cdot c_d L, \end{aligned}$$

where c_d and L are defined in Theorem 1.

Proposition 2. *Suppose that the same assumptions hold as in Theorem 2. Then we have as $n \to \infty$,*
Case 2A: *we have the same result as in Case 1A of Proposition 1,*
Case 2B: $\frac{\tau_n}{nh}\sum_{i=1}^{n} K_i \eta_i \rho_q'(V_i) = o_p(1)$,
Case 2C:

$$\frac{\tau_n}{nh}\sum_{i=1}^{n} K_i \eta_i \rho_q'(V_i) \xrightarrow{d} f(x_0)\left(\begin{array}{c} 1 \\ 0 \end{array}\right)\sigma_{\alpha\beta}(c_+^{1/(\alpha\beta)}C_q^+ L^+ + c_-^{1/(\alpha\beta)}C_q^- L^-),$$

where $\sigma_{\alpha\beta}$, $C_q^{\pm}$, and $L^{\pm}$ are defined in Theorem 2.

Proposition 3. *Suppose that the same assumptions hold as in Theorem 3. Then we have the same result as in Case 1A of Proposition 1.*

Proposition 4. *Suppose that the Assumptions V, K, H, Z1-3, and X1-2 hold. Then for any fixed* θ*, we have as* $n \to \infty$*,*

$$\frac{\tau_n^2}{nh}\sum_{i=1}^n K_i(\rho_q(V_i^* - \tau_n^{-1}\eta_i\theta) - \rho_q(V_i^*))$$
$$= \frac{1}{2}\theta^T \begin{pmatrix} 1 & 0 \\ 0 & \kappa_2 \end{pmatrix} \theta f_V(0|x_0)f(x_0) - \Big(\frac{\tau_n}{nh}\sum_{i=1}^n K_i\eta_i\rho_q'(V_i^*)\Big)^T\theta + o_p(1).$$

The bias term in Proposition 5 below is negligible in Cases 1B, 2B, and 2C since $\tau_n/\sqrt{nh} \to 0$ in the cases.

Proposition 5. *Suppose that the Assumptions V, K, H, Z1-3, and X1-2 hold. Then we have as* $n \to \infty$*,*

$$\frac{\tau_n}{nh}\sum_{i=1}^n K_i\eta_i\rho_q'(V_i^*)$$
$$= \frac{\tau_n}{nh}\sum_{i=1}^n K_i\eta_i\rho_q'(V_i) + \frac{\tau_n}{2\sqrt{nh}}\begin{pmatrix} c_h^{5/2}\kappa_2 u''(x_0)f_V(0|x_0)f(x_0) \\ 0 \end{pmatrix} + o_p(1).$$

Now we prove Theorem 1 as in Fan et al.(1994) and Hall et al.(2002) by adapting the method of Pollard (1991) to nonparametric regression. Theorems 2-3 can be established in the same way by applying Propositions 2-3, respectively and the proofs are omitted.

Proof of Theorem 1. Recall that $\tau_n/\sqrt{nh} = 1$ in Case 1A and $\tau_n/\sqrt{nh} = o(1)$ in Case 1B. Equation (4) is equivalent to

$$\hat\theta = \mathrm{argmin}_{\theta\in\mathbb{R}^2}\frac{\tau_n^2}{nh}\sum_{i=1}^n K_i(\rho_q(V_i^* - \tau_n^{-1}\eta_i^T\theta) - \rho_q(V_i^*)). \qquad (13)$$

By Propositions 4-5, we have for any fixed $\theta \in \mathbb{R}^2$,

$$\frac{\tau_n^2}{nh}\sum_{i=1}^n K_i(\rho_q(V_i^* - \tau_n^{-1}\eta_i\theta) - \rho_q(V_i^*)) \qquad (14)$$
$$= \frac{1}{2}\theta^T \begin{pmatrix} 1 & 0 \\ 0 & \kappa_2 \end{pmatrix} \theta f_V(0|x_0)f(x_0) - \Big(\frac{\tau_n}{nh}\sum_{i=1}^n K_i\eta_i\rho_q'(V_i)\Big)^T\theta$$
$$-\frac{\tau_n}{2\sqrt{nh}}(c_h^{5/2}\kappa_2 u''(x_0)f_V(0|x_0)f(x_0), 0)\theta + o_p(1).$$

As in Pollard (1991), Fan et al. (1994), and Hall et al. (2002), the convexity lemma implies that (14) holds uniformly on $\{|\theta| < M\}$ for any positive M.

We consider the RHS of (14). Proposition 1 implies that

$$\frac{\tau_n}{nh}\sum_{i=1}^{n} K_i \eta_i \rho_q'(V_i) = O_p(1). \tag{15}$$

Combining (15), $\tau_n/\sqrt{nh} = O(1)$, the uniformity of (14), and the convexity of the objective function in (13), we conclude that $|\hat{\theta}| = O_p(1)$ by appealing to the standard argument.

By using $|\hat{\theta}| = O_p(1)$ and the uniformity of (14) again, we obtain

$$\begin{aligned}\hat{\theta} &= \frac{1}{f_V(0|x_0)f(x_0)}\begin{pmatrix} 1 & 0 \\ 0 & \kappa_2 \end{pmatrix}^{-1} \\ &\quad\times\Big\{\frac{\tau_n}{nh}\sum_{i=1}^{n} K_i \eta_i \rho_q'(V_i) + \frac{\tau_n}{2\sqrt{nh}}(c_h^{5/2}\kappa_2 u''(x_0) f_V(0|x_0) f(x_0), 0)^T\Big\} + o_p(1).\end{aligned} \tag{16}$$

The results of the theorem follow from (16) and Proposition 1. Hence the proof of the theorem is complete.

5. Proofs of Propositions 1-5

We describe Lemmas 1-3 before we prove Propositions 1-5. The proofs of the lemmas are postponed to Section 6. We introduce some more notation for Lemmas 1-3.

Define $B_{\xi,s}(\tilde{Z}_{i,s-1})$ and $B_{\xi,\infty}(v)$ for $\xi \in [-C_K, C_K]$ by

$$B_{\xi,s}(\tilde{Z}_{i,s-1}) = \mathrm{E}\{B_\xi(Z_i)|\mathcal{G}_{i-s}\} \quad \text{and} \quad B_{\xi,\infty}(v) = \mathrm{E}\{B_\xi(Z_1 + v)\},$$

where $B_\xi(z)$ is uniformly bounded in ξ and will be specified in the proofs of Propositions 1-5. When $B_\xi(z)$ does not depend on ξ, we omit ξ in $B_\xi(z)$.

Next we define $o_{m,r}(a_n)$ for $r \geq 1$ by

$$W_\xi = o_{m,r}(a_n) \Leftrightarrow \|a_n^{-1}W_\xi\|_r = o(1) \text{ uniformly in } \xi. \tag{17}$$

The definition of $O_{m,r}(a_n)$ is obvious from (17).

Recall that $A_\xi(i) = f_1(x_0+\xi h|\mathcal{F}_{i-1})$ and $\mathrm{E}\{A_\xi(i)\} = f(x_0+\xi h)$. Hereafter we omit "as $n \to \infty$".

Lemma 1. *Suppose that Assumptions X1-2 and Z1-3 hold in Case 1.*
(i) There exists $1 < r < \alpha$ *s.t.*

$$\begin{aligned} & \frac{1}{n}\sum_{i=1}^{n} A_\xi(i) B_{\xi,1}(\tilde{Z}_{i,0}) \qquad (18) \\ = & \ (f(x_0+\xi h) + o_{m,r}(1))\mathrm{E}\{B_{\xi,1}(\tilde{Z}_{1,0})\} + \frac{1}{n}B'_{\xi,\infty}(0)\sum_{i=1}^{n} A_\xi(i)\tilde{Z}_{i,0} \\ & +o_{m,r}(n^{-\beta+1/\alpha}). \end{aligned}$$

(ii) When Assumption X3 with $p = \alpha$ *or X4 holds, we can replace* $A_\xi(i)$ *in the RHS of (18) with* $\mathrm{E}\{A_\xi(i+j)\} = f(x_0+\xi h)$.

It is easy to see that $\mathrm{E}\{|n^{-1}\sum_{i=1}^{n} A_\xi(i)\tilde{Z}_{i,0}|^r\} = o(1)$ for any $1 < r < \alpha$.

When we use an assumption similar to Assumption X4 instead of Assumption X5 in Lemma 2(ii) below, we have to assume that $2/(\alpha\beta) - 1 < \gamma_X$ to obtain the same result.

Lemma 2. *Suppose that Assumptions X1-2 and Z1-3 hold in Case 2.*
(i) There exists $1 < r < \alpha\beta$ *s.t.*

$$\begin{aligned} & \frac{1}{n}\sum_{i=1}^{n} A_\xi(i) B_{\xi,1}(\tilde{Z}_{i,0}) \qquad (19) \\ = & \ (f(x_0+\xi h) + o_{m,r}(1))\mathrm{E}\{B_{\xi,1}(\tilde{Z}_{1,0})\} \\ & +\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{\infty} A_\xi(i+j)(B_{\xi,j}(c_j\zeta_i) - \mathrm{E}\{B_{\xi,j}(c_j\zeta_i)\}) + o_{m,r}(n^{-1+1/(\alpha\beta)}). \end{aligned}$$

In addition, for any $1 < r < \alpha\beta$,

$$\mathrm{E}\{|\sum_{j=1}^{\infty} A_\xi(i+j)(B_{\xi,j}(c_j\zeta_i) - \mathrm{E}\{B_{\xi,j}(c_j\zeta_i)\})|^r\} < C$$

uniformly in ξ *and* i.
(ii) When Assumption X3 with $\alpha\beta < p$ *or X5 holds, we can replace* $A_\xi(i+j)$ *in the RHS of (19) with* $\mathrm{E}\{A_\xi(i+j)\} = f(x_0+\xi h)$. *Besides, when* $B_\xi(z) = B(z)$ *for some function* $B(z)$, *we have*

$$n^{-1/(\alpha\beta)}\sum_{i=1}^{n}\sum_{j=1}^{\infty}(B_j(c_j\zeta_i) - \mathrm{E}\{B_j(c_j\zeta_i)\}) \xrightarrow{d} \sigma_{\alpha\beta}(c_+^{1/(\alpha\beta)}C_B^+L^+ + c_-^{1/(\alpha\beta)}C_B^-L^-),$$

where $C_B^{\pm} = \int_0^{\infty}(B_\infty(\pm v) - B_\infty(0))v^{-(1+1/\beta)}dv$. *See Theorem 2 for the definitions* $\sigma_{\alpha\beta}$ *and* $L^{\pm}$.

Lemma 3. *Suppose that Assumptions X1-2 and Z1-3 hold in Case 3. Then we have*

$$\frac{1}{n}\sum_{i=1}^{n} A_\xi(i)B_{\xi,1}(\tilde{Z}_{i,0}) = (f(x_0+\xi h) + o_{m,p}(1))\mathrm{E}\{B_{\xi,1}(\tilde{Z}_{1,0})\} + O_{m,2}(n^{-1/2}).$$

Now we begin to prove Propositions 1-5.

Proof of Proposition 1. We follow Wu and Mielniczuk (2002), Mielniczuk and Wu (2004), and Honda (2009a). We consider only the first element. The second element can be treated in the same way.

Set

$$T_i = K_i\rho_q'(V_i) - \mathrm{E}\{K_i\rho_q'(V_i)|\mathcal{S}_{i-1}\}.$$

Note that $|T_i| \le C$ and that

$$\begin{aligned}
&\frac{1}{nh}\sum_{i=1}^{n}\mathrm{E}\{T_i^2|\mathcal{S}_{i-1}\} \qquad (20)\\
&= \frac{1}{n}\sum_{i=1}^{n}\int\int K^2(\xi)f_1(x_0+\xi h|\mathcal{F}_{i-1})(\rho_q'(V(x_0+\xi h,z)))^2 g_0(z-\tilde{Z}_{i,0})d\xi dz\\
&= \frac{\nu_0}{n}\sum_{i=1}^{n} f_1(x_0|\mathcal{F}_{i-1})\int(\rho_q'(V(x_0,z)))^2 g_0(z-\tilde{Z}_{i,0})dz + o_p(1)\\
&\xrightarrow{p} \nu_0 f(x_0)q(1-q)
\end{aligned}$$

We used the monotonicity of $V(x,z)$ in z, Assumption X1, and the ergodic theorem in (20). Therefore by the martingale central limit theorem,

$$\frac{\tau_n}{nh}\sum_{i=1}^{n}T_i \quad \begin{cases} \xrightarrow{d} N(0, f(x_0)q(1-q)\nu_0) & \text{in Case1A,}\\ = o_p(1) & \text{in Case1B.}\end{cases} \qquad (21)$$

Next we deal with $\mathrm{E}\{K_i\rho_q'(V_i)|\mathcal{S}_{i-1}\}$. Since

$$\begin{aligned}
&\frac{1}{h}\mathrm{E}\{K_i\rho_q'(V_i)|\mathcal{S}_{i-1}\} \qquad (22)\\
&= \int K(\xi)\Big\{f_1(x_0+\xi h|\mathcal{F}_{i-1})\int\rho_q'(V(x_0+\xi h,z))g_0(z-\tilde{Z}_{i,0})dz\Big\}d\xi,
\end{aligned}$$

we apply Lemma 1 with $B_\xi(z) = \rho_q'(V(x_0+\xi h, z)) = \rho_q'(V(x_0, z))$ and

$$B_{\xi,1}(\tilde{Z}_{i,0}) = \int \rho_q'(V(x_0+\xi h, z))g_0(z-\tilde{Z}_{i,0})dz = \int \rho_q'(V(x_0, z))g_0(z-\tilde{Z}_{i,0})dz.$$

Notice that

$$\mathrm{E}\{B_{\xi,1}(\tilde{Z}_{i,0})\} = 0 \text{ and } B_{\xi,\infty}'(0) = -\int \rho_q'(V(x_0, z))g_\infty'(z)dz. \tag{23}$$

From Lemma 1(ii) and (23), we have in Case 1B that

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^n A_\xi(i)B_{\xi,1}(\tilde{Z}_{i,0}) \qquad (24)\\ &= -f(x_0+\xi h)\int \rho_q'(V(x_0, z))g_\infty'(z)dz\frac{1}{n}\sum_{i=1}^n \tilde{Z}_{i,0} + o_{m,r}(n^{-\beta+1/\alpha}). \end{aligned}$$

From Jensen's inequality w.r.t. $\int \cdot K(\xi)d\xi$, (22), and (24), we obtain

$$\begin{aligned} &\frac{1}{nh}\sum_{i=1}^n \mathrm{E}\{K_i\rho_q'(V_i)|\mathcal{S}_{i-1}\} \qquad (25)\\ &= -f(x_0)\int \rho_q'(V(x_0, z))g_\infty'(z)dz\frac{1}{n}\sum_{i=1}^n \tilde{Z}_{i,0} + o_p(n^{-\beta+1/\alpha}). \end{aligned}$$

We can proceed in a similar way in Case 1A by employing Lemma 1(i). Thus by (25) and the definition of τ_n,

$$\frac{\tau_n}{nh}\sum_{i=1}^n \mathrm{E}\{K_i\rho_q'(V_i)|\mathcal{S}_{i-1}\} \quad \begin{cases} = o_p(1) & \text{in Case1A,} \\ = -f(x_0)\int \rho_q'(V(x_0, z))g_\infty'(z)dz & \\ \quad \times \frac{\tau_n}{n}\sum_{i=1}^n \tilde{Z}_{i,0} + o_p(1) & \text{in Case1B.} \end{cases} \tag{26}$$

The desired result follows from (21), (26), and Kasahara and Maejima (1988). Hence the proof is complete.

Proof of Proposition 2. We define T_i as in the proof of Proposition 1 and T_i can be treated in the same way as in the proof of Proposition 1. Then we have

$$\frac{\tau_n}{nh}\sum_{i=1}^n T_i \quad \begin{cases} \stackrel{d}{\to} N(0, f(x_0)q(1-q)\nu_0) & \text{in Case2A,} \\ = o_p(1) & \text{in Case2B, C.} \end{cases} \tag{27}$$

Next we deal with $\frac{1}{h}\mathrm{E}\{K_i\rho'_q(V_i)|\mathcal{S}_{i-1}\}$ by applying Lemma 2 as in the proof of Proposition 1.

By Lemma 2(i),

$$\begin{aligned}
&\frac{1}{n}\int K(\xi)A_\xi(i)B_{\xi,1}(\tilde{Z}_{i,0})d\xi \\
&= \frac{1}{n}\sum_{i=1}^{n}\int K(\xi)\Big\{\sum_{j=1}^{\infty}A_\xi(i+j)(B_{\xi,j}(c_j\zeta_i)-\mathrm{E}\{B_{\xi,j}(c_j\zeta_i)\})\Big\}d\xi \\
&\qquad +o_p(n^{-1+1/(\alpha\beta)}).
\end{aligned}$$

From the latter half of Lemma 2(i), we have for any $1 < r < \alpha\beta$,

$$\frac{1}{n}\sum_{i=1}^{n}\int K(\xi)A_\xi(i)B_{\xi,1}(\tilde{Z}_{i,0})d\xi = O_p(n^{-1+1/r}). \tag{28}$$

Finally we consider the case where Assumption X3 with $\alpha\beta < p$ or X5 holds. Then Lemma 2(ii), the monotonicity of $V(x,z)$ in z, and Jensen's inequality w.r.t. $\int \cdot K(\xi)d\xi$ yield that

$$\begin{aligned}
&\frac{1}{n}\int K(\xi)A_\xi(i)B_{\xi,1}(\tilde{Z}_{i,0})d\xi \\
&= \frac{1}{n}\int K(\xi)f(x_0+\xi h)d\xi\sum_{i=1}^{n}\sum_{j=1}^{\infty}(B_j(c_j\eta_i)-\mathrm{E}\{B_j(c_j\eta_i)\})+o_p(n^{-1+1/(\alpha\beta)}),
\end{aligned}$$

where $B(z)=\rho'_q(V(x_0,z))$. The convergence in distribution follows from the latter half of Lemma 2(ii) with

$$B_\infty(v)=\int(q-I(z+v<m_q))g_\infty(z)dz = q-G_\infty(m_q-v).$$

Consequently we have

$$\frac{\tau_n}{nh}\sum_{i=1}^{n}\mathrm{E}\{K_i\rho'_q(V_i)|\mathcal{S}_{i-1}\}\begin{cases} = o_p(1) & \text{in Case2A, B,} \\ \xrightarrow{d} \sigma_{\alpha\beta}(c_+^{1/(\alpha\beta)}C_q^+L^+ \\ \qquad +c_-^{1/(\alpha\beta)}C_q^-L^-) & \text{in Case2C.}\end{cases} \tag{29}$$

The desired result follows from (27) and (29). Hence the proof of the lemma is complete.

Proof of Proposition 3. We can proceed as in the proofs of Propositions 1-2 by appealing to Lemma 3. The details are omitted.

Proof of Proposition 4. We establish Proposition 4 by employing Lemmas 1-3. Set

$$S_\theta(X_i, Z_i) = \rho_q(V_i^* - \tau_n^{-1}\eta_i^T\theta) - \rho_q(V_i^*) + \tau_n^{-1}\eta_i^T\theta\rho_q'(V_i^*).$$

Since $|V_i^* - V_i| \le Ch^2$ and $\tau_n = O(h^{-2})$, we have

$$|S_\theta(X_i, Z_i)| \le C|\tau_n^{-1}\eta_i^T\theta| I(|V_i| \le C\tau_n^{-1}|\theta|).$$

Letting

$$T_i = K_i S_\theta(X_i, Z_i) - \mathrm{E}\{K_i S_\theta(X_i, Z_i)|\mathcal{S}_{i-1}\},$$

we have

$$\frac{\tau_n^2}{nh}\sum_{i=1}^n T_i = o_p(1) \tag{30}$$

because

$$\begin{aligned}\mathrm{E}\Big\{\Big(\frac{\tau_n^2}{nh}\sum_{i=1}^n T_i\Big)^2\Big\} &\le C\frac{\tau_n^2|\theta|^2}{(nh)^2}\sum_{i=1}^n \mathrm{E}\{K_i^2 I(|V_i| \le C\tau_n^{-1}|\theta|)\} \\ &\le C\frac{\tau_n|\theta|^3}{nh} \to 0.\end{aligned}$$

Next we deal with $\mathrm{E}\{K_i S_\theta(X_i, Z_i)|\mathcal{S}_{i-1}\}$, which is written as

$$\begin{aligned}&\frac{\tau_n^2}{h}\mathrm{E}\{K_i S_\theta(X_i, Z_i)|\mathcal{S}_{i-1}\} \\ &= \int K(\xi)\Big\{f_1(x_0 + \xi h|\mathcal{F}_{i-1})\tau_n^2 \int S_\theta(x_0 + \xi h, z) g_0(z - \tilde{Z}_{i,0})dz\Big\}d\xi.\end{aligned} \tag{31}$$

We take $B_\xi(z) = \tau_n^2 S_\theta(x_0 + \xi h, z)$ for Lemmas 1-3 and have

$$\mathrm{E}\{B_\xi(Z_i)\} = \frac{1}{2}((1, \xi)\theta)^2 f_V(0|x_0) + o(1) \quad \text{uniformly in } \xi.$$

Note that $B_\xi(z)$ is not uniformly bounded in ξ. However, $B_{\xi,1}(z)$ is uniformly bounded in ξ. Therefore we should apply Lemma 1-3 with $\tilde{Z}_{i,0}$ and

$B_{\xi,1}(z)$ replaced by $\tilde{Z}_{i,1}$ and $B_{\xi,2}(z)$. Then we have for some $1 < r$,

$$\frac{1}{n}\sum_{i=1}^{n} A_\xi(i)B_{\xi,2}(\tilde{Z}_{i,1})) = \frac{1}{2}((1,\xi)\theta)^2 f_V(0|x_0)f(x_0) + o_{m,r}(1), \tag{32}$$

$$\frac{1}{n}\sum_{i=1}^{n} A_\xi(i)(B_{\xi,1}(\tilde{Z}_{i,0}) - B_{\xi,2}(\tilde{Z}_{i,1})) = O_{m,2}(n^{-1/2}). \tag{33}$$

By (31)-(33),

$$\frac{\tau_n^2}{nh}\mathrm{E}\{K_i S_\theta(X_i, Z_i)|\mathcal{S}_{i-1}\} = \frac{1}{2}\theta^T \begin{pmatrix} 1 & 0 \\ 0 & \kappa_2 \end{pmatrix} \theta f_V(0|x_0)f(x_0) + o_p(1). \tag{34}$$

The desired result follows from (30) and (34). Hence the proof of the proposition is complete.

Proof of Proposition 5. we can prove Proposition 5 in the same way as Proposition 4 by setting

$$T_i = K_i(\rho_q'(V_i^*) - \rho_q'(V_i)) - \mathrm{E}\{K_i(\rho_q'(V_i^*) - \rho_q'(V_i))|\mathcal{S}_{i-1}\}$$

and

$$B_\xi(z) = \tau_n(\rho_q'(V^*(x_0 + \xi h, z)) - \rho_q'(V(x_0 + \xi h, z))).$$

The details are omitted.

6. Technical lemmas

We establish Lemmas 1-3 in this section. We state Lemmas 4-6 before the proof of Lemma 1, Lemmas 7-8 before the proof of Lemma 2, and Lemma 9 before the proof of Lemma 3, respectively. The proofs of Lemmas 4-9 are given at the end of this section.

Lemma 4 below is essentially Lemma 4.1 of Koul and Surgailis (2001) and Lemma 4.1 deals with empirical distribution functions. We just describe the necessary changes in the proof of Lemma 4.

Lemma 4. *Suppose that Assumptions X1-2 and Z1-3 hold in Case 1. Then there exists* $1 < r < \alpha$ *s.t.*

$$\frac{1}{n}\sum_{i=1}^{n} A_\xi(i)(B_{\xi,1}(\tilde{Z}_{i,0}) - \mathrm{E}\{B_{\xi,1}(\tilde{Z}_{i,0})\} - B'_{\xi,\infty}(0)\tilde{Z}_{i,0}) = o_{m,r}(n^{-\beta+1/\alpha}).$$

Lemma 5. *Suppose that Assumptions X1-2, X3 with* $p = \alpha$, *and Z1-3 hold in Case 1. Then there exists* $1 < r < \alpha$ *s.t.*

$$\frac{1}{n}\sum_{i=1}^{n}(A_\xi(i) - \mathrm{E}\{A_\xi(i)\})B'_{\xi,\infty}(0)\tilde{Z}_{i,0} = o_{m,r}(n^{-\beta+1/\alpha}).$$

Lemma 6. *Suppose that Assumptions X1-2, X4, and Z1-3 hold in Case 1. Then there exists* $1 < r < \alpha$ *s.t.*

$$\frac{1}{n}\sum_{i=1}^{n}(A_\xi(i) - \mathrm{E}\{A_\xi(i)\})B'_{\xi,\infty}(0)\tilde{Z}_{i,0} = o_{m,r}(n^{-\beta+1/\alpha}).$$

Proof of Lemma 1. From Lemmas 4-6, we have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} A_\xi(i)(B_{\xi,1}(\tilde{Z}_{i,0}) - \mathrm{E}\{B_{\xi,1}(\tilde{Z}_{i,0})\}) \\
&\quad = \frac{1}{n}A_\xi(i)B'_{\xi,\infty}(0)\tilde{Z}_{i,0} + o_{m,r_1}(n^{-\beta+1/\alpha}) \\
&\quad = \frac{1}{n}\mathrm{E}\{A_\xi(i)\}B'_{\xi,\infty}(0)\tilde{Z}_{i,0} + o_{m,r_1}(n^{-\beta+1/\alpha}) + o_{m,r_2}(n^{-\beta+1/\alpha}),
\end{aligned}$$

where r_1 is from Lemma 1, r_2 is from Lemma 2 or 3, and $1 < r_1, r_2 < \alpha$. We set $r = r_1 \wedge r_2$ and apply (9) to $\mathrm{E}\{B_{\xi,1}(\tilde{Z}_{1,0})\}\sum_{i=1}^{n} A_\xi(i)$. Hence the proof of Lemma 1 is complete.

Lemma 7 below is essentially proved for $1 < \alpha < 2$ and for $0 < \alpha \le 1$ in Surgailis (2001) and Honda (2009b), respectively. We just outline the proof later in this section.

Lemma 7. *Suppose that Assumptions X1-2 and and Z1-3 hold in Case 2. Then there exists* $1 < r < \alpha\beta$ *s.t.*

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} A_\xi(i)(B_{\xi,1}(\tilde{Z}_{i,0}) - \mathrm{E}\{B_{\xi,1}(\tilde{Z}_{i,0})\}) \\
&\quad = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{\infty} A_\xi(i+j)(B_{\xi,j}(c_j\zeta_i) - \mathrm{E}\{B_{\xi,j}(c_j\zeta_i)\}) + o_{m,r}(n^{-1+1/(\alpha\beta)}).
\end{aligned}$$

Lemma 8. *Suppose that Assumptions X1-2 and and Z1-3 hold in Case 2. In addition, Assumption X3 with* $\alpha\beta < p$ *or X5 holds. Then there exists* $1 < r < \alpha\beta$ *s.t.*

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n} A_\xi(i)(B_{\xi,1}(\tilde{Z}_{i,0}) - \mathrm{E}\{B_{\xi,1}(\tilde{Z}_{i,0})\})\\ &= \frac{1}{n}f(x_0+\xi h)\sum_{i=1}^{n}\sum_{j=1}^{\infty}(B_{\xi,j}(c_j\zeta_i) - \mathrm{E}\{B_{\xi,j}(c_j\zeta_i)\}) + o_{m,r}(n^{-1+1/(\alpha\beta)}).\end{aligned}$$

Proof of Lemma 2.
(i) The former half of (i) follows from Lemma 7 and (9).

Next by following Lemma 3.1 of Surgailis (2002) and Proposition 2.3 of Honda (2009b), we can demonstrate that given $\{\epsilon_i\}$,

$$\mathrm{limsup}_{|z|\to\infty}|z|^{-1/\beta}\Big|\sum_{j=1}^{\infty}A_\xi(i+j)(B_{\xi,j}(c_jz) - \mathrm{E}\{B_{\xi,j}(c_j\zeta_i)\}\Big| \le C,$$

uniformly in ξ and i and C is independent of $\{\epsilon_i\}$. This implies that

$$\mathrm{limsup}_{z\to\infty}z^{\alpha\beta}\mathrm{P}\Big(\Big|\sum_{j=1}^{\infty}A_\xi(i+j)(B_{\xi,j}(c_j\zeta_i) - \mathrm{E}\{B_{\xi,j}(c_j\zeta_i)\})\Big| > z\Big) \le C, \quad (35)$$

uniformly in ξ and i. The latter half of (i) follows from (35)
(ii) The desired result follows from (i), Lemma 8, and Proposition 2.3 of Honda (2009b).

Lemma 9 below is almost given in Pipiras and Taqqu (2003) and we just give an outline of the proof at the end of this section.

Lemma 9. *Suppose that Assumptions X1-2 and Z1-3 hold in Case 3. Then we have*

$$\frac{1}{n}\sum_{i=1}^{n}A_\xi(i)(B_{\xi,1}(\tilde{Z}_{i,0}) - \mathrm{E}\{B_{\xi,1}(\tilde{Z}_{i,0})\}) = O_{m,2}(n^{-1/2}).$$

Proof of Lemma 3. We can verify Lemma 3 in the same way as Lemmas 1-2 by using Lemma 9 . The details are omitted.

We give the proofs of Lemmas 4-9 here.

Proof of Lemma 4. We only present necessary changes to the proof of Lemma 4.1 of Koul and Surgailis (2001). We define $H(z)$ in (4.1) there by

$$H(z) = A_\xi(t)(B_{\xi,1}(z) - \mathrm{E}\{B_{\xi,1}(\tilde{Z}_{t,0})\} - B'_\infty(0)z).$$

Then φ_n, $U_{t,s}$, $U_{t,s}^{(0)}$, $U_{t,s}^{(1)}$, $U_{t,s}^{(2)}$, and $U_{t,s}^{(3)}$ are given by

$$\begin{aligned}
\varphi_n &= \sum_{t=1}^{n} A_\xi(t) H(\tilde{Z}_{t,0}),\\
U_{t,s} &= A_\xi(t)(\mathrm{E}\{B_{\xi,1}(\tilde{Z}_{t,0})|\mathcal{G}_{t-s}\} - \mathrm{E}\{B_{\xi,1}(\tilde{Z}_{t,0})|\mathcal{G}_{t-s-1}\} - B'_{\xi,\infty}(0)c_s\zeta_{t-s})\\
&= A_\xi(t)\Big\{B_{\xi,s}(c_s\zeta_{t-s} + \tilde{Z}_{t,s}) - \int B_{\xi,s}(c_s u + \tilde{Z}_{t,s})dG_0(u)\\
&\qquad -B'_{\xi,\infty}(0)c_s\zeta_{t-s}\Big\},\\
U_{t,s}^{(0)} &= U_{t,s},\\
U_{t,s}^{(1)} &= A_\xi(t)\Big\{B_{\xi,s}(c_s\zeta_{t-s} + \tilde{Z}_{t,s}) - \int B_{\xi,s}(c_s u + \tilde{Z}_{t,s})dG_0(u)\\
&\qquad -B'_{\xi,s}(\tilde{Z}_{t,s})c_s\zeta_{t-s}\Big\},\\
U_{t,s}^{(2)} &= A_\xi(t)\{-c_s\zeta_{t-s}(B'_{\xi,\infty}(0) - B'_{\xi,\infty}(\tilde{Z}_{t,s}))\},\\
U_{t,s}^{(3)} &= A_\xi(t)\{-c_s\zeta_{t-s}(B'_{\xi,\infty}(\tilde{Z}_{t,s}) - B'_{\xi,s}(\tilde{Z}_{t,s}))\}.
\end{aligned}$$

We can treat $A_\xi(t)$ as if it were a constant because of the independence of $\{\zeta_i\}$ and $\{\epsilon_i\}$.

The treatment of $U_{t,s}^{(0)}$ is trivial. We consider $U_{t,s}^{(1)}$, $U_{t,s}^{(2)}$, and $U_{t,s}^{(3)}$. We write $U_{t,s}^{(1)}$ as

$$U_{t,s}^{(1)} = W_{t,s}^{(1)} - W_{t,s}^{(2)},$$

where

$$\begin{aligned}
W_{t,s}^{(1)} &= A_\xi(t)\int\Big[\int_{-c_s u}^{-c_s\zeta_{t-s}}\Big\{\int B_\xi(w)g'_s(w + z - \tilde{Z}_{t,s})dw\Big\}dz\Big]dG_0(u),\\
W_{t,s}^{(2)} &= A_\xi(t)\int\Big[\int_{-c_s u}^{-c_s\zeta_{t-s}}\Big\{\int B_\xi(w)g'_s(w - \tilde{Z}_{t,s})dw\Big\}dz\Big]dG_0(u).
\end{aligned}$$

In the above expressions, the integrals $\int_x^y \cdot dw$ in Koul and Surgailis (2001) are replaced with $\int_{\mathbb{R}} B_\xi(w) \cdot dw$ here. However, $(1+|w|)^{-(1+\gamma_z)}$ from Assumption Z3 appears in the integrals and this change does not affect the integrability and the argument about $U_{t,s}^{(1)}$ at all.

We can treat $U_{t,s}^{(2)}$ and $U_{t,s}^{(3)}$ similarly because they are written as

$$\begin{aligned} U_{t,s}^{(2)} &= A_\xi(t) c_s \zeta_{t-s} \int B_\xi(z)(g'_\infty(z) - g'_\infty(z - \tilde{Z}_{t,s}))dz, \\ U_{t,s}^{(3)} &= A_\xi(t) c_s \zeta_{t-s} \int B_\xi(z)(g'_\infty(z - \tilde{Z}_{t,s}) - g'_s(z - \tilde{Z}_{t,s}))dz. \end{aligned}$$

Hence the desired result follows from the arguments of Lemma 4.1 of Koul and Surgailis (2001).

Proof of Lemma 5. Write

$$A_\xi(i) - \mathrm{E}\{A_\xi(i)\} = \sum_{j=1}^{\infty} D_{i,j}, \tag{36}$$

where $D_{i,j} = \mathrm{E}\{A_\xi(i)|\mathcal{F}_{i-j}\} - \mathrm{E}\{A_\xi(i)|\mathcal{F}_{i-j-1}\}$. Then we have for any $1 < r < \alpha$,

$$\|D_{i,j}\| \le \theta_r(j) \quad \text{and} \quad \sum_{j=1}^{\infty} \theta_r(j) < \infty. \tag{37}$$

By using (36) and rearranging the summation, we get

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^{n} \tilde{Z}_{i,0}(A_\xi(i) - \mathrm{E}\{A_\xi(i)\}) \qquad (38) \\ &= \frac{1}{n}\sum_{i=1}^{n} \tilde{Z}_{i,0} \sum_{j=1}^{\infty} D_{i,j} = \frac{1}{n}\sum_{l=-\infty}^{n-1} \sum_{j=(1-l)\vee 1}^{n-l} \tilde{Z}_{j+l,0} D_{j+l,j}. \end{aligned}$$

Thus from (37) and (38), we have as in Wu et al.(2010),

$$\begin{aligned} &\mathrm{E}\Big\{\Big|n^{-1+\beta-1/\alpha} \sum_{i=1}^{n} \tilde{Z}_{i,0}(A_\xi(i) - \mathrm{E}\{A_\xi(i)\})\Big|^r\Big\} \qquad (39) \\ &\le Cn^{-r+r\beta-r/\alpha} \sum_{l=-\infty}^{n-1} \Big(\sum_{j=(1-l)\vee 1}^{n-l} \theta_r(j)\Big)^r \mathrm{E}\{|\tilde{Z}_{i,0}|^r\} \\ &= O(n^{-r+1+r\beta-r/\alpha}) \quad \text{uniformly in } \xi. \end{aligned}$$

We can choose $1 < r < \alpha$ satisfying $-r+1+r\beta - r/\alpha < 0$. Hence the proof of Lemma 5 is complete.

Proof of Lemma 6. Fix $1 < r < \alpha$ satisfying $r\beta > 1$ temporarily. We specify r later in the proof. Setting $c_j = 0$ for $j < 0$, we have

$$\sum_{i=1}^{n}(A_\xi(i) - \mathrm{E}\{A_\xi(i)\})\tilde{Z}_{i,0} = \sum_{j=-\infty}^{n-1} \zeta_j \sum_{i=1}^{n}(A_\xi(i) - \mathrm{E}\{A_\xi(i)\})c_{i-j}. \tag{40}$$

From (40) and the von Bahr and Esseen inequality, we have

$$\begin{aligned} &\mathrm{E}\Big\{|\sum_{i=1}^{n}(A_\xi(i) - \mathrm{E}\{A_\xi(i)\})\tilde{Z}_{i,0}|^r\Big\} \qquad (41)\\ &\le\ C \sum_{j=-\infty}^{n-1} \mathrm{E}\{|\zeta_1|^r\}\mathrm{E}\Big\{\Big|\sum_{i=1}^{n}(A_\xi(i) - \mathrm{E}\{A_\xi(i)\})c_{i-j}\Big|^r\Big\} \\ &\le\ C \sum_{j=-\infty}^{n-1} \mathrm{E}\{|\zeta_1|^r\}\Big(\mathrm{E}\Big\{\Big|\sum_{i=1}^{n}(A_\xi(i) - \mathrm{E}\{A_\xi(i)\})c_{i-j}\Big|^2\Big\}\Big)^{r/2}. \end{aligned}$$

We will give an upper bound of (41) by evaluating

$$\mathrm{E}\Big\{\Big|\sum_{i=1}^{n}(A_\xi(i) - \mathrm{E}\{A_\xi(i)\})c_{i-j}\Big|^2\Big\}. \tag{42}$$

When $j \le 0$, Assumption X4 implies that (42) is bounded from above by

$$\sum_{i=1}^{n}\mathrm{Var}(A_\xi(i))c_{i-j}^2 + C\sum_{1\le i_1<i_2\le n}|i_1 - i_2|^{-\gamma_x}|c_{i_1-j}c_{i_2-j}|. \tag{43}$$

It is easy to see that

$$\begin{aligned} &\sum_{j=-\infty}^{0}\Big(\sum_{i=1}^{n}\mathrm{Var}(A_\xi(i))c_{i-j}^2\Big)^{r/2} \qquad (44)\\ &\le\ C\sum_{j=-\infty}^{0}\sum_{i=1}^{n}\frac{1}{(i+|j|)^{r\beta}}\ \le C\sum_{i=1}^{n}i^{1-r\beta}\ \le\ Cn^{2-r\beta}. \end{aligned}$$

We have $n^{2-r\beta} < n^{r-r\beta+r/\alpha}$ when $2/(1+1/\alpha) < r < \alpha$.

In order to evaluate the last expression of (41), we fix δ_1 and δ_2 satisfying $0 < \delta_1 < 1/(1+\gamma_x)$ and $0 < \delta_2 < ((1-\beta)(1+\gamma_x/2)^{-1}) \wedge (r/\alpha)$, respectively.

When $j \leq -n$, the second term of (43) is bounded from above by

$$\sum_{i_2 - i_1 \geq n^{\delta_1}} |i_1 - i_2|^{-\gamma_x} |c_{i_1-j} c_{i_2-j}| + \sum_{0 < i_2 - i_1 < n^{\delta_1}} |i_1 - i_2|^{-\gamma_x} |c_{i_1-j} c_{i_2-j}|$$
$$\leq C(n^{2-\gamma_x\delta_1}|j|^{-2\beta} + n^{1+\delta_1}|j|^{-2\beta}).$$

Taking the summation of $j \leq -n$ in the RHS of (41), we get

$$\begin{aligned} &\sum_{j \leq -n} ((n^{2-\gamma_x\delta_1} + n^{1+\delta_1})|j|^{-2\beta})^{r/2} \\ &\leq \; Cn^{r - r\gamma_x\delta_1/2} \sum_{j \leq -n} |j|^{-r\beta} \; \leq \; Cn^{r(1-\beta)+1-r\gamma_x\delta_1/2}. \end{aligned} \tag{45}$$

We have $n^{r(1-\beta)+1-r\gamma_x\delta_1/2} < n^{r-r\beta+r/\alpha}$ when $1/(1/\alpha + \gamma_x\delta_1/2) < r < \alpha$.

When $-n < j \leq -n^{\delta_2}$, the second term of (43) is bounded from above by

$$\sum_{i_2 - i_1 \geq n^{\delta_2/2}} |i_1 - i_2|^{-\gamma_x} |c_{i_1-j} c_{i_2-j}| + \sum_{0 < i_2 - i_1 < n^{\delta_2/2}} |i_1 - i_2|^{-\gamma_x} |c_{i_1-j} c_{i_2-j}|$$
$$\leq C(n^{-\gamma_x\delta_2/2+2-2\beta} + n^{1-\beta+\delta_2/2}).$$

Taking the summation of $-n < j \leq -n^{\delta_2}$ in the RHS of (41), we get

$$\sum_{-n < j \leq -n^{\delta_2}} (n^{-\gamma_x\delta_2/2+2-2\beta} + n^{1-\beta+\delta_2/2})^{r/2} \leq Cn^{r(1-\beta)+1-r\gamma_x\delta_2/4}. \tag{46}$$

We have $n^{r(1-\beta)+1-r\gamma_x\delta_2/4} < n^{r-r\beta+r/\alpha}$ when $1/(1/\alpha + \gamma_x\delta_2/4) < r < \alpha$.

When $-n^{\delta_2} < j \leq 0$, the second term of (43) is bounded from above by $Cn^{2-2\beta}$. The summation of $-n^{\delta_2} < j \leq 0$ in the RHS of (41) is bounded from above by $Cn^{\delta_2 + r(1-\beta)}$, which is smaller than $n^{r-r\beta+r/\alpha}$.

We can deal with the case of $0 < j < n$ in almost the same way and the summation of $0 < j < n$ in the RHS of (41) is bounded from above by $C(n^{-r\gamma_x\delta_2/4+r(1-\beta)+1} + n^{\delta_2+r(1-\beta)} + n)$, which is smaller than $n^{r-r\beta+r/\alpha}$ when $(1/(1/\alpha + 1 - \beta)) \vee (1/(1/\alpha + \gamma_x\delta_2/4)) < r < \alpha$. From this and (44)-(46), we obtain the desired result by choosing r satisfying

$$\frac{2}{1/\alpha + 1} \vee \frac{1}{1/\alpha + 1 - \beta} \vee \frac{1}{1/\alpha + \gamma_x\delta_2/4} \vee \frac{1}{1/\alpha + \gamma_x\delta_1/2} < r < \alpha.$$

Hence the proof of Lemma 6 is complete.

We need Lemmas 10-11 to deal with Case 2. The lemmas follow from (3.35) and (3.41) of Pipiras and Taqqu (2003) and some calculation. We omit the proofs. In the lemmas below, $l_1(x)$, $l_2(x)$, and $l_3(x)$ are slow varying functions and necessary only when $\alpha = 1$.

Lemma 10. *Suppose that Assumptions Z1-2 hold and $\alpha\beta > 1$. Then we have for any $j \geq 0$,*

$$\mathrm{P}(|c_j\zeta_1| \geq 1) \leq C|c_j|^\alpha l_1(1/|c_j|) \text{ and } \mathrm{P}(|\tilde{Z}_{1,j}| \geq 1) \leq Cl_2(j)(1+j)^{1-\alpha\beta}.$$

Lemma 11. *Suppose that Assumptions Z1-2 hold in Case 2. Then there exists C_γ for any $\alpha\beta < \gamma \leq 2$ s.t. for any $j \geq 0$,*

$$\begin{aligned}
\mathrm{E}\{|c_j\zeta_1|^\gamma I(|c_j\zeta_1| < 1)\} &\leq C_\gamma l_3(j)|c_j|^\alpha, \\
\mathrm{E}\{|\tilde{Z}_{1,j}|^\gamma I(|\tilde{Z}_{1,j}| < 1)\} &\leq C_\gamma l_3(j)(1+j)^{1-\alpha\beta}.
\end{aligned}$$

Proof of Lemma 7. Set as in Honda (2009b),

$$\begin{aligned}
S_n &= \sum_{i=1}^n A_\xi(i)(B_{\xi,1}(\tilde{Z}_{i,0}) - \mathrm{E}\{B_{\xi,1}(\tilde{Z}_{i,0})\}), \\
T_n &= \sum_{i=1}^n A_\xi(i)\sum_{j=1}^\infty (B_{\xi,j}(c_j\zeta_{i-j}) - \mathrm{E}\{B_{\xi,j}(c_j\zeta_{i-j})\}), \\
W_n &= \sum_{i=1}^n \sum_{j=1}^\infty A_\xi(i+j)(B_{\xi,j}(c_j\zeta_i) - \mathrm{E}\{B_{\xi,j}(c_j\zeta_i)\}).
\end{aligned}$$

By employing Lemmas 10-11 and following the arguments in Honda (2009b), we can demonstrate that there exists $\alpha\beta < r_1 < 2 \wedge (2\alpha\beta - 1)$ s.t.

$$\mathrm{E}\{|S_n - T_n|^{r_1}\} \leq C(n^{-2\alpha\beta+2+r_1} + n) \tag{47}$$

and that there exists $1 < r_2 < \alpha\beta$ s.t.

$$\mathrm{E}\{|T_n - W_n|^{r_2}\} \leq Cn^{-\alpha\beta+r_2+1}. \tag{48}$$

Note that the arguments apply to the cases of both $1 < \alpha < 2$ and $0 < \alpha \leq 1$ and that the inequalities (47) and (48) hold uniformly in ξ. Hence we obtain the desired result by setting $r = r_1 \wedge r_2$.

Proof of Lemma 8. We only give the proof to the case where $\alpha \neq 1$ and Assumption X5 holds. The other cases can be similarly treated.

Define T_n' and W_n' by

$$\begin{aligned} T_n' &= \mathrm{E}\{A_\xi(1)\}\sum_{i=1}^{n}\sum_{j=1}^{\infty}(B_{\xi,j}(c_j\zeta_{i-j}) - \mathrm{E}\{B_{\xi,j}(c_j\zeta_{i-j})\}) \\ W_n' &= \mathrm{E}\{A_\xi(1)\}\sum_{i=1}^{n}\sum_{j=1}^{\infty}(B_{\xi,j}(c_j\zeta_i) - \mathrm{E}\{B_{\xi,j}(c_j\zeta_i)\}). \end{aligned}$$

Then as in the proof of Lemma 7, we have uniformly in ξ,

$$\mathrm{E}\{|T_n' - W_n'|^{r_2}\} \leq Cn^{-\alpha\beta + r_2 + 1}. \tag{49}$$

We define Δ_1, Δ_2, and Δ_{3i} in the RHS of (50) below to evaluate the difference between T_n and T_n'.

$$\begin{aligned} & T_n - T_n' \qquad (50) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n-i}(A_\xi(i+j) - \mathrm{E}\{A_\xi(i+j)\})(B_{\xi,j}(c_j\zeta_i) - \mathrm{E}\{B_{\xi,j}(c_j\zeta_i)\}) \\ &\qquad + \sum_{i=1}^{n}\sum_{j=i}^{\infty}(A_\xi(i) - \mathrm{E}\{A_\xi(i)\})(B_{\xi,j}(c_j\zeta_{i-j}) - \mathrm{E}\{B_{\xi,j}(c_j\zeta_{i-j})\}) \\ &= \Delta_1 + \Delta_2 = \sum_{i=1}^{n}\Delta_{3i} + \Delta_2. \end{aligned}$$

The proof of Proposition 2.2 of Honda (2009b) implies that uniformly in ξ,

$$\mathrm{E}\{|\Delta_2|^{r_2}\} \leq Cn^{-\alpha\beta + r_2 + 1}. \tag{51}$$

We consider Δ_1 by exploiting the fact that Δ_{3i}, $i = 1, 2, \ldots, n$, are mutually independent.

Provided that there exists $\alpha\beta < r_3 < 2$ s.t. uniformly in i and ξ,

$$\mathrm{E}\{|\Delta_{3i}|^{r_3}\} \leq C, \tag{52}$$

we have from the independence of $\{\Delta_{3i}\}$,

$$\mathrm{E}\{|\Delta_1|^{r_3}\} \leq Cn. \tag{53}$$

Then we have from (51) and (53),

$$T_n - T_n' = o_{m,r_2}(n^{1/(\alpha\beta)}). \qquad (54)$$

Hence the desired result follows from (47), (49), and (54). Thus we have only to verify (52).

Write

$$\begin{aligned}
\Delta_{3i} &= \sum_{j=1}^{n-i}\Big\{\sum_{l=1}^{\infty}(\mathrm{E}\{A_\xi(i+j)|\mathcal{F}_{i+j-l}\} - \mathrm{E}\{A_\xi(i+j)|\mathcal{F}_{i+j-l-1}\})\Big\} \qquad (55)\\
&\qquad \times(B_{\xi,j}(c_j\zeta_i) - \mathrm{E}\{B_{\xi,j}(c_j\zeta_i)\})\\
&= \sum_{l=-n+1}^{\infty}\sum_{j=1\vee(1-i-l)}^{n-i}(\mathrm{E}\{A_\xi(i+j)|\mathcal{F}_{-l}\} - \mathrm{E}\{A_\xi(i+j)|\mathcal{F}_{-l-1}\})\\
&\qquad \times(B_{\xi,j}(c_j\zeta_i) - \mathrm{E}\{B_{\xi,j}(c_j\zeta_i)\})
\end{aligned}$$

and fix r_3 satisfying

$$\alpha\beta < r_3 < 2 \wedge \frac{\alpha\beta(\alpha\beta-1)}{\alpha\beta - 1 - \delta_x\alpha\beta/2} \wedge r_x \wedge \frac{\alpha\beta}{1-\delta_x}.$$

In addition we get from Lemmas 10-11,

$$\|B_{\xi,j}(c_j\zeta_i) - \mathrm{E}\{B_{\xi,j}(c_j\zeta_i)\}\|_{r_3} \le Cj^{-\alpha\beta/r_3}. \qquad (56)$$

Thus from (55) and (56), we have uniformly in i and ξ,

$$\begin{aligned}
&\mathrm{E}\{|\Delta_{3i}|^{r_3}\}\\
&\le C\sum_{l=-n+1}^{\infty}\Big(\sum_{j=1\vee(1-i-l)}^{n-i}\|B_{\xi,j}(c_j\zeta_i) - \mathrm{E}\{B_{\xi,j}(c_j\zeta_i)\}\|_{r_3}\theta_{r_3}(i+j+l)\Big)^{r_3}\\
&\le C\sum_{l=-i}^{\infty}\Big(\sum_{j=1\vee(1-i-l)}^{n-i}\frac{1}{j^{\alpha\beta/r_3}}\cdot\frac{1}{(i+j+l)^{\delta_x+1/(\alpha\beta)}}\Big)^{r_3}\\
&\qquad +C\sum_{l=-n+1}^{-i-1}\Big(\sum_{s=0}^{n+l}\frac{1}{(s-l-i)^{\alpha\beta/r_3}}\cdot\frac{1}{(s+1)^{\delta_x+1/(\alpha\beta)}}\Big)^{r_3}\\
&\le C\sum_{l=-i}^{\infty}\frac{1}{(i+l+1)^{r_3/(\alpha\beta)}}\Big(\sum_{j=1\vee(1-i-l)}^{n-i}\frac{1}{j^{\delta_x+\alpha\beta/r_3}}\Big)^{r_3}
\end{aligned}$$

$$+C\sum_{l=-n+1}^{-i-1}\frac{1}{(-l-i)^{\alpha\beta-r_3(1-1/(\alpha\beta)-\delta_x/2)}}\Big(\sum_{s=0}^{n+l}\frac{1}{(s+1)^{1+\delta_x/2}}\Big)^{r_3}$$
$$\leq \; C.$$

Hence (52) is established and the proof of Lemma 8 is complete.

Proof of Lemma 9. We just outline the proof. We define $U_{i,j}$ by the RHS of (57) below.

$$A_\xi(i)(B_{\xi,1}(\tilde{Z}_{i,0})-\mathrm{E}\{B_{\xi,1}(\tilde{Z}_{i,0})\}) \tag{57}$$
$$= \sum_{j=1}^{\infty}A_\xi(i)(\mathrm{E}\{B_{\xi,1}(\tilde{Z}_{i,0})|\mathcal{G}_{i-j}\}-\mathrm{E}\{B_{\xi,1}(\tilde{Z}_{i,0})|\mathcal{G}_{i-j-1}\}) = \sum_{j=1}^{\infty}A_\xi(i)U_{i,j}.$$

Then we have

$$U_{i,j}=\int B_\xi(\eta)\Big\{\int(g_j(\eta-c_j\zeta_{i-j}-\tilde{Z}_{i,j})-g_j(\eta-c_j\zeta-\tilde{Z}_{i,j}))dG_0(\zeta)\Big\}d\eta.$$

Besides, $\int B_\xi(\eta)g_j(\eta-v)d\eta$ and the derivative are uniformly bounded by Assumption Z3. Thus we have

$$\Big|\int B_\xi(\eta)(g_j(\eta-c_j\zeta_{i-j}-\tilde{Z}_{i,j})-g_j(\eta-c_j\zeta-\tilde{Z}_{i,j}))d\eta\Big| \tag{58}$$
$$\leq C1\wedge|c_j(\zeta_{i-j}-\zeta)|.$$

Inequalities (58) above and (3.35) and (3.36) of Pipiras and Taqqu (2003) yield

$$\mathrm{E}\{(A_\xi(i)U_{i,j})^2\}\leq C|c_j|^\alpha. \tag{59}$$

By the definition of $U_{i,j}$ and (59),

$$\mathrm{E}\Big\{\Big(\sum_{i=1}^{n}A_\xi(i)\sum_{j=1}^{\infty}U_{i,j}\Big)^2\Big\} \tag{60}$$
$$\leq \; C\sum_{i=1}^{n}\sum_{j=1}^{\infty}\sum_{j'=1}^{\infty}(\mathrm{E}\{(A_\xi(i)U_{i,j})^2\})^{1/2}(\mathrm{E}\{(A_\xi(i')U_{i',j'})^2\})^{1/2} \leq Cn,$$

where $i'=i-j+j'$.

Hence the desired result follows from (57) and (60).

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Table 2: α =1.1(γ =0.75)

β		0.9		1.3		1.7		∞	
h		0.2	0.4	0.2	0.4	0.2	0.4	0.2	0.4
0.0	mean	0.072	0.163	0.032	0.078	0.017	0.067	0.018	0.066
	bias	0.072	0.163	0.032	0.078	0.017	0.067	0.018	0.066
	mse	N/A	N/A	N/A	N/A	*0.362	*0.333	0.059	0.032
	madv	1.927	1.923	0.683	0.661	0.317	0.286	0.191	0.141
0.6	mean	1.183	1.332	1.018	1.179	1.028	1.169	1.028	1.170
	bias	0.204	0.353	0.039	0.200	0.048	0.190	0.049	0.190
	mse	N/A	N/A	N/A	N/A	*0.677	*0.466	0.090	0.078
	madv	1.928	1.912	0.764	0.736	0.362	0.355	0.224	0.229

Table 3: α =1.2(γ =0.75)

β		0.9		1.3		1.7		∞	
h		0.2	0.4	0.2	0.4	0.2	0.4	0.2	0.4
0.0	mean	0.013	0.015	0.055	0.105	0.002	0.051	0.016	0.066
	bias	0.013	0.015	0.055	0.105	0.002	0.051	0.016	0.066
	mse	N/A	N/A	N/A	N/A	*0.644	*1.139	0.060	0.033
	madv	1.505	1.460	0.534	0.509	0.285	0.255	0.192	0.145
0.6	mean	0.971	1.172	1.074	1.217	1.003	1.154	1.031	1.173
	bias	-0.008	0.193	0.094	0.238	0.024	0.175	0.052	0.194
	mse	N/A	N/A	N/A	N/A	*2.299	*1.651	0.092	0.081
	madv	1.496	1.603	0.582	0.571	0.334	0.328	0.228	0.232

Table 4: α =1.3(γ =0.75)

β		0.9		1.3		1.7		∞	
h		0.2	0.4	0.2	0.4	0.2	0.4	0.2	0.4
0.0	mean	0.050	0.098	0.027	0.074	0.012	0.066	0.013	0.063
	bias	0.050	0.098	0.027	0.074	0.012	0.066	0.013	0.063
	mse	N/A	N/A	*2.580	*2.778	0.116	0.086	0.061	0.034
	madv	0.921	0.908	0.423	0.396	0.260	0.223	0.195	0.147
0.6	mean	1.065	1.214	1.035	1.157	1.026	1.175	1.028	1.170
	bias	0.086	0.235	0.056	0.177	0.047	0.196	0.049	0.191
	mse	N/A	N/A	*2.094	*4.499	0.163	0.145	0.087	0.081
	madv	0.956	0.946	0.450	0.458	0.296	0.295	0.227	0.232

Table 5: α =1.4(γ =0.75)

β		0.9		1.3		1.7		∞	
h		0.2	0.4	0.2	0.4	0.2	0.4	0.2	0.4
0.0	mean	0.025	0.073	0.009	0.059	0.013	0.064	0.014	0.065
	bias	0.025	0.073	0.009	0.059	0.013	0.064	0.014	0.065
	mse	N/A	N/A	*1.199	*1.333	0.096	0.068	0.063	0.035
	madv	0.751	0.737	0.357	0.33	0.244	0.206	0.199	0.149
0.6	mean	1.030	1.174	1.022	1.165	1.028	1.170	1.024	1.172
	bias	0.051	0.194	0.043	0.186	0.049	0.191	0.045	0.193
	mse	N/A	N/A	*1.049	*1.116	0.131	0.120	0.090	0.082
	madv	0.779	0.779	0.389	0.387	0.279	0.275	0.229	0.233

Table 6: α =1.5(γ =0.75)

β		0.9		1.3		1.7		∞	
h		0.2	0.4	0.2	0.4	0.2	0.4	0.2	0.4
0.0	mean	0.034	0.085	0.007	0.060	0.016	0.066	0.012	0.064
	bias	0.034	0.085	0.007	0.060	0.016	0.066	0.012	0.064
	mse	N/A	N/A	0.238	0.213	0.091	0.064	0.063	0.035
	madv	0.613	0.597	0.316	0.288	0.237	0.199	0.198	0.149
0.6	mean	1.049	1.194	1.023	1.167	1.033	1.178	1.031	1.176
	bias	0.070	0.215	0.043	0.188	0.054	0.199	0.052	0.197
	mse	N/A	N/A	0.280	0.275	0.129	0.120	0.088	0.083
	madv	0.636	0.638	0.347	0.344	0.275	0.275	0.230	0.236

Table 7: $\alpha = 1.1 (\gamma = 1.25)$

β		0.9		1.3		1.7		∞	
h		0.2	0.4	0.2	0.4	0.2	0.4	0.2	0.4
0.0	mean	0.064	0.158	0.017	0.069	0.015	0.067	0.019	0.066
	bias	0.064	0.158	0.017	0.069	0.015	0.067	0.019	0.066
	mse	N/A	N/A	N/A	N/A	*0.368	*0.334	0.058	0.032
	madv	1.825	1.806	0.681	0.653	0.309	0.277	0.189	0.143
0.6	mean	1.171	1.325	1.052	1.188	1.027	1.173	1.025	1.172
	bias	0.192	0.346	0.073	0.209	0.048	0.194	0.046	0.193
	mse	N/A	N/A	N/A	N/A	*0.467	*0.421	0.071	0.072
	madv	1.874	1.886	0.701	0.695	0.330	0.334	0.209	0.223

Table 8: $\alpha = 1.2 (\gamma = 1.25)$

β		0.9		1.3		1.7		∞	
h		0.2	0.4	0.2	0.4	0.2	0.4	0.2	0.4
0.0	mean	-0.125	-0.094	0.049	0.107	-0.005	0.050	0.015	0.065
	bias	-0.125	-0.094	0.049	0.107	-0.005	0.050	0.015	0.065
	mse	N/A	N/A	N/A	N/A	*1.304	*1.031	0.058	0.032
	madv	1.611	1.579	0.510	0.492	0.282	0.247	0.190	0.143
0.6	mean	0.939	1.129	1.058	1.212	1.019	1.161	1.027	1.175
	bias	-0.040	0.150	0.079	0.232	0.040	0.182	0.048	0.196
	mse	N/A	N/A	N/A	N/A	*0.374	*0.600	0.076	0.076
	madv	1.543	1.519	0.523	0.537	0.292	0.305	0.217	0.228

Table 9: $\alpha = 1.3 (\gamma = 1.25)$

β		0.9		1.3		1.7		∞	
h		0.2	0.4	0.2	0.4	0.2	0.4	0.2	0.4
0.0	mean	0.050	0.097	0.007	0.079	0.017	0.066	0.015	0.065
	bias	0.050	0.097	0.007	0.079	0.017	0.066	0.015	0.065
	mse	N/A	N/A	*8.059	*3.151	0.109	0.080	0.060	0.033
	madv	0.913	0.899	0.440	0.390	0.249	0.214	0.194	0.145
0.6	mean	1.062	1.208	1.037	1.182	1.027	1.173	1.026	1.175
	bias	0.083	0.228	0.058	0.203	0.048	0.194	0.047	0.195
	mse	N/A	N/A	*2.900	*2.471	0.126	0.126	0.075	0.075
	madv	0.921	0.917	0.434	0.431	0.270	0.278	0.215	0.226

Table 10: α=1.4(γ=1.25)

β		0.9		1.3		1.7		∞	
h		0.2	0.4	0.2	0.4	0.2	0.4	0.2	0.4
0.0	mean	0.017	0.071	0.006	0.059	0.011	0.063	0.016	0.066
	bias	0.017	0.071	0.006	0.059	0.011	0.063	0.016	0.066
	mse	N/A	N/A	*1.320	*0.983	0.090	0.065	0.060	0.034
	madv	0.746	0.732	0.353	0.321	0.237	0.201	0.193	0.147
0.6	mean	1.033	1.186	1.022	1.168	1.028	1.174	1.024	1.173
	bias	0.054	0.206	0.043	0.189	0.049	0.195	0.045	0.194
	mse	N/A	N/A	*0.872	*0.891	0.110	0.109	0.077	0.076
	madv	0.766	0.764	0.366	0.368	0.259	0.265	0.220	0.229

Table 11: α=1.5(γ=1.25)

β		0.9		1.3		1.7		∞	
h		0.2	0.4	0.2	0.4	0.2	0.4	0.2	0.4
0.0	mean	0.040	0.089	0.008	0.060	0.017	0.068	0.015	0.066
	bias	0.040	0.089	0.008	0.060	0.017	0.068	0.015	0.066
	mse	N/A	N/A	0.231	0.200	0.086	0.060	0.061	0.034
	madv	0.605	0.588	0.309	0.279	0.231	0.194	0.197	0.148
0.6	mean	1.051	1.196	1.03	1.174	1.028	1.175	1.031	1.178
	bias	0.072	0.217	0.051	0.195	0.049	0.196	0.052	0.199
	mse	N/A	N/A	0.264	0.251	0.104	0.104	0.077	0.078
	madv	0.613	0.618	0.328	0.333	0.255	0.260	0.222	0.230