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Limiting Distribution of the Score Statistic under Moderate Deviation from a Unit Root in MA(1)

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LIMITING DISTRIBUTION OF THE SCORE STATISTIC UNDER MODERATE DEVIATION FROM A UNIT ROOT IN MA(1)

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Abstract

This paper derives the asymptotic distribution of Tanaka's score statistic under moderate deviation from a unit root in a moving average model of order one or MA(1). We classify the limiting distribution into three types depending on the order of deviation. In the fastest case, the convergence order of the asymptotic distribution continuously changes from the invertible process to the unit root. In the slowest case, the limiting distribution coincides with the invertible process in a distributional sense. This implies that these cases share an asymptotic property. The limiting distribution in the intermediate case provides the boundary property between the fastest and slowest cases.

1 INTRODUCTION

It is a well-known that in an autoregressive process (AR) with a unit root, the asymptotic distribution of the ordinary least squares estimator is a functional of Brownian motion. Conversely, in a stationary process, the limiting distribution is normal. This difference highlights that unit root and stationary processes have different asymptotic properties. To investigate this distinction, Giraitis and Phillips (2006) and Phillips and Magdalinos (2007) have considered the moderate deviation class of the local-to-unity, where the moderate deviation AR process is the local-to-unity AR process that has a distance between the AR coefficient and unity larger than $O(T^{-1})$ where T is the sample size. Giraitis and Phillips (2006) and Phillips and Magdalinos (2007) have shown that the martingale central limit theorem (CLT) for the ordinary least squares estimator in stationary regions holds and its convergence order continuously changes from the unit root to the stationary process. Phillips and Magdalinos (2007) has also considered

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explosive regions. Cointegration theory associated with moderate deviation has been also developed by Magdalinos and Phillips (2009) and Kurozumi and Hayakawa (2009).

In this paper, we develop the asymptotic theory of the moderate deviation from a unit root in a moving average model of order one or MA(1). An MA(1) process is said to be noninvertible when the process has a unit root. If not, the MA(1) model is invertible. Importantly, the estimators and test statistics in a noninvertible process behave quite differently from the invertible MA(1) model. For example, the CLT holds for the maximum likelihood estimator under an invertible process, but does not hold in a noninvertible process because it has a probability mass at unity (see Sargan and Bhargava (1983) and Anderson and Takemura (1986)).

In this paper, we consider the fractional form of moderate deviation. That is, the MA(1) coefficient is defined by $\rho_T = 1 - c/T^{\alpha}$ where c is a positive constant and $\alpha \in (0, 1)$, such that the distance between the coefficient and unity is $O(T^{-\alpha})$. We derive the asymptotic distribution of the normalized score statistic in MA(1) to investigate the asymptotic property of the score test statistic where the largest MA root is moderately smaller than unity. We show that the asymptotic distribution is classified into three types, namely, (i) $1/2 < \alpha < 1$, (ii) $\alpha = 1/2$ and (iii) $0 < \alpha < 1/2$. In the fastest case, (i) $1/2 < \alpha < 1$, the convergence order depends on α and continuously changes with respect to α, while in the smallest order case, (iii) $0 < \alpha < 1/2$, the convergence order is independent of α and the asymptotic distribution coincides with that under invertibility when the constant coefficient is less than one. This feature indicates that the largest deviation and invertibility have a common asymptotic property. Moreover, in the intermediate case, (ii) $\alpha = 1/2$, the asymptotic distribution converges to that of (iii) $0 < \alpha < 1/2$ when we take the sequential limit as $c \to \infty$ after $T \to \infty$.

The remainder of this paper is organized as follows. Section 2 introduces the model and provides the assumptions. In Section 3, we derive the asymptotic distribution under the assumption of normal, independent and identically distributed (N.I.D.) shocks and report the numerical results. In Section 4, we relax the N.I.D. assumption and extend the disturbance term to a linearly dependent process.

Throughout the paper, we employ the following notation: $\to_p$ and $\Rightarrow$ to denote convergence in probability and weak convergence, respectively, as the sample size T goes to infinity, and $\mathrm{P}(\cdot)$ and $\mathcal{L}(\cdot)$ to denote the probability measure and law, respectively.

2 MODEL AND ASSUMPTION

We consider the following MA(1) model with moderate deviation from a unit root:

$$y_t = u_t - \rho_T u_{t-1} \qquad (t = 1, \ldots, T),$$

where $\rho_T = 1 - c/T^{\alpha}$, $\alpha \in (0,1)$ and $T \in \mathbb{N}$. We assume that c is a positive constant, such that $\rho_T \in [-1,1]$, in order to satisfy the identification condition of ρ_T and $u_t \sim$ N.I.D.$(0, \sigma^2)$. We assume $\sigma^2 = 1$ without loss of generality as the score statistic is scale invariant.

Tanaka (1990b) has considered the following MA unit root-testing problem:

$$H_0 : \rho_T = 1 \quad \text{v.s} \quad H_1 : \rho_T = 1 - \frac{c}{T}.$$

In the testing problem, moderate deviation corresponds to the local alternative $H_1 : \rho_T = 1 - c/T^{\alpha}$. He derived the score test statistic S_T that is equivalent to the locally best invariant test statistic (LBI) defined by:

$$S_T = \frac{y'\Omega^{-2}y}{y'\Omega^{-1}y},$$

where $y = (y_1, \ldots, y_T)'$ and

$$\Omega = \begin{pmatrix} 2 & -1 & & \mathbf{0} \\ -1 & 2 & -1 & \\ & \ddots & \ddots & -1 \\ \mathbf{0} & & -1 & 2 \end{pmatrix} : (T \times T).$$

Note that Ω is a variance–covariance matrix of y under a unit root process with $\sigma^2 = 1$ and its eigenvalues $\{\lambda_{t,T}\}$ are given by:

$$\lambda_{t,T} = 4 \sin^2 \frac{t\pi}{2(T+1)} \qquad (t = 1, \ldots, T),$$

as in Sargan and Bhargava (1983) and Anderson and Takemura (1986).

3 ASYMPTOTIC DISTRIBUTION OF THE SCORE STATISTIC

3.1 Derivation of the Asymptotic Distribution

Tanaka (1990b) has derived the following asymptotic distribution by the eigenvalue approach for a (near) MA unit root process. The asymptotic distribution of the normalized score statistic in a unit root case ($\rho_T = 1$) is given by:

$$\mathcal{L}\left(\frac{S_T}{T}\right) \to \mathcal{L}\left(\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} Z_n^2\right),$$

and in a near unit root case ($\rho_T = 1 - c/T$) by:

$$\mathcal{L}\left(\frac{S_T}{T}\right) \to \mathcal{L}\left(\sum_{n=1}^{\infty} \left(\frac{1}{n^2\pi^2} + \frac{c^2}{n^4\pi^4}\right) Z_n^2\right), \tag{1}$$

as $T \to \infty$, where $\{Z_n\} \sim$ N.I.D.$(0,1)$. Details of the eigenvalue approach are found in Tanaka (1996). In this section, we derive the asymptotic distribution of the normalized score statistic S_T in the moderate deviation case. For this purpose, we first derive the asymptotic distributions of the normalized numerator and denominator with the following two lemmas.

Lemma 1. *As $T \to \infty$ with $\rho_T = 1 - c/T^\alpha$ where $\alpha \in (0,1)$, the asymptotic distribution of the normalized numerator of S_T is given by:*

$$\mathcal{L}\left(\frac{y'\Omega^{-2}y}{T^{4-2\alpha}}\right) \to \mathcal{L}\left(\sum_{n=1}^{\infty} \frac{c^2}{n^4\pi^4} Z_n^2\right).$$

Lemma 1 gives the asymptotic distribution of the normalized numerator of S_T. This result shows the first term in the near unit root case disappears in (1). The normalized order continuously changes from T^4 to T^2 as α changes from 0 to 1. Continuous change of the convergence rate is also observed in the moderate deviation AR(1) model in Giraitis and Phillips (2006) and Phillips and Magdalinos (2007).

Lemma 2. *As $T \to \infty$ with $\rho_T = 1 - c/T^\alpha$ where $\alpha \in (0,1)$, the asymptotic distribution of the denominator of S_T is classified into three types. Each asymptotic distribution is given by:*

(i) $1/2 < \alpha < 1$

$$\frac{y'\Omega^{-1}y}{T} \to_p 1,$$

(ii) $\alpha = 1/2$

$$\mathcal{L}\left(\frac{y'\Omega^{-1}y}{T}\right) \to \mathcal{L}\left(1 + \sum_{n=1}^{\infty} \frac{c^2}{n^2\pi^2} Z_n^2\right),$$

(iii) $0 < \alpha < 1/2$

$$\mathcal{L}\left(\frac{y'\Omega^{-1}y}{T^{2-2\alpha}}\right) \to \mathcal{L}\left(\sum_{n=1}^{\infty} \frac{c^2}{n^2\pi^2} Z_n^2\right).$$

The convergence order in the moderate deviation case also continuously changes from T^2 to T as α goes from 0 to 1/2. In case (i), the denominator converges in probability to 1, so as to unit root and near unit root in Tanaka (1990). Conversely, the additional term appears in case (ii) while only the nondegenerate term remains in case (iii).

Combining Lemmas 1 and 2 and the continuous mapping theorem implies the following theorem.

Theorem 1. *As* $T \to \infty$ *with* $\rho_T = 1 - c/T^\alpha$ *where* $\alpha \in (0, 1)$*, the asymptotic distribution of the normalized score statistic* S_T *is classified into three types. Each asymptotic distribution is given by:*

(i)$1/2 < \alpha < 1$

$$\mathcal{L}\left(\frac{S_T}{T^{3-2\alpha}}\right) \to \mathcal{L}\left(\sum_{n=1}^{\infty} \frac{c^2}{n^4\pi^4} Z_n^2\right), \tag{2}$$

(ii)$\alpha = 1/2$

$$\mathcal{L}\left(\frac{S_T}{T^2}\right) \to \mathcal{L}\left(\frac{\sum_{n=1}^{\infty} \frac{c^2}{n^4\pi^4} Z_n^2}{1 + \sum_{n=1}^{\infty} \frac{c^2}{n^2\pi^2} Z_n^2}\right), \tag{3}$$

(iii)$0 < \alpha < 1/2$

$$\mathcal{L}\left(\frac{S_T}{T^2}\right) \to \mathcal{L}\left(\frac{\sum_{n=1}^{\infty} \frac{1}{n^4\pi^4} Z_n^2}{\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} Z_n^2}\right). \tag{4}$$

The convergence order under moderate deviation also continuously changes from T to T^2 as α goes from 1 to 0. In case (iii), the parameter c disappears as c is canceled out because of the fractional form of the asymptotic distribution.

As expected, the convergence order depends on α when α takes values between 1/2 and 1. Of particular interest is that the convergence order is free from α for $0 < \alpha \leq 1/2$. Moreover, the asymptotic distribution in case (ii) converges to that in case (iii) in probability when we take the sequential limit $c \to \infty$. That is, as $c \to \infty$:

$$
\begin{aligned}
\frac{\sum_{n=1}^{\infty} \frac{c^2}{n^4\pi^4} Z_n^2}{1 + \sum_{n=1}^{\infty} \frac{c^2}{n^2\pi^2} Z_n^2} &= \frac{\sum_{n=1}^{\infty} \frac{1}{n^4\pi^4} Z_n^2}{\frac{1}{c^2} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} Z_n^2} \\
&\to_p \frac{\sum_{n=1}^{\infty} \frac{1}{n^4\pi^4} Z_n^2}{\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} Z_n^2}.
\end{aligned}
$$

We summarize this argument in the following corollary.

Corollary 1. *As $T \to \infty$ followed by $c \to \infty$, the asymptotic distribution in case (ii) $\alpha = 1/2$ converges to that in case (iii) $0 < \alpha < 1/2$ in probability.*

We can also derive the asymptotic distribution of the normalized S_T under an invertible process.

Corollary 2. *As $T \to \infty$ in the invertible case, that is ρ_T is constant in $(-1, 1)$, the asymptotic distribution of the normalized score statistic S_T is given by:*

$$
\mathcal{L}\left(\frac{S_T}{T^2}\right) \to \mathcal{L}\left(\frac{\sum_{n=1}^{\infty} \frac{Z_n^2}{n^4\pi^4}}{\sum_{n=1}^{\infty} \frac{Z_n^2}{n^2\pi^2}}\right).
$$

Theorem 1 and Corollary 2 show that the asymptotic distributions in case (iii) and invertibility coincide in distribution. From this fact, we can conclude that moderate deviation in case (iii) and invertibility share an asymptotic property. In the next subsection, we undertake numerical computation of the asymptotic distributions.

3.2 Numerical Calculation

In this subsection, we numerically calculate the distribution and density functions. For this purpose, we derive the limiting characteristic function (c.f.) of the normalized score statistic S_T. See Tanaka (1996) for details of the numerical procedure involved.

First, we give the limiting c.f. of S_T in case (i) $1/2 < \alpha < 1$.

Theorem 2. *As $T \to \infty$, the limiting c.f. $\phi_1(\theta)$ of the normalized score statistic S_T in case (i)*

$1/2 < \alpha < 1$ *is given by:*

$$\phi_1(\theta) = \left(\frac{\sinh \sqrt{c\sqrt{2i\theta}} \sinh \sqrt{-c\sqrt{2i\theta}}}{\sqrt{c\sqrt{2i\theta}}\sqrt{-c\sqrt{2i\theta}}} \right)^{-1/2}.$$

Using the c.f. derived, we can calculate the distribution by inverting $\phi_1(\theta)$. Figure 1 depicts the asymptotic distribution functions of the normalized score statistics S_T given in (2) for various values of $c = 1, \ldots, 6$. These distribution functions move to the right as c becomes large. This phenomenon is observed in case (ii).

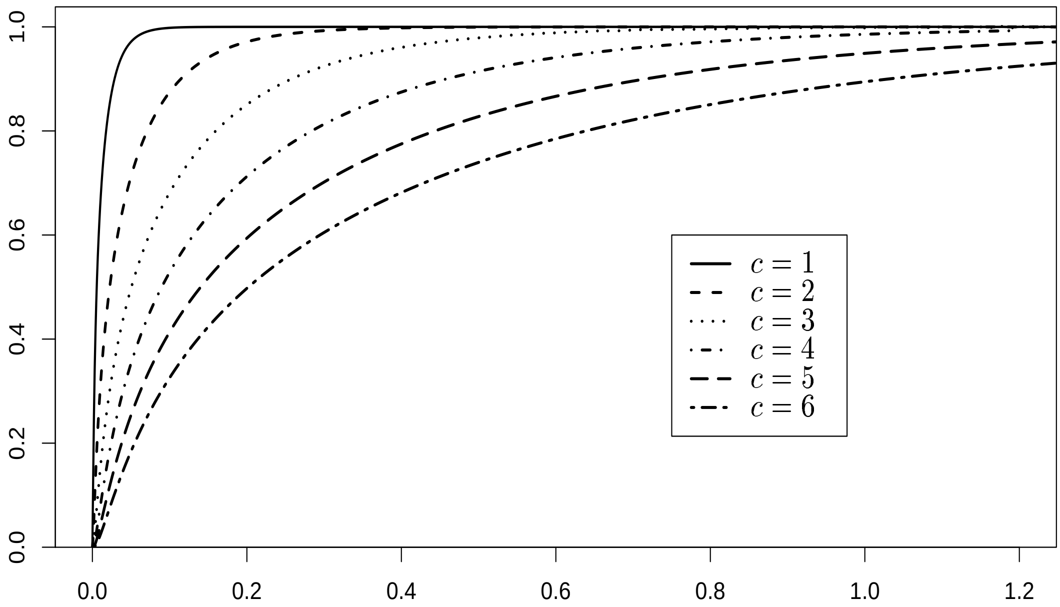

Figure 1: Distribution functions: (i) $1/2 < \alpha < 1$

Given the limiting distributions in cases (ii) and (iii) given in (3) and (4) take a fractional form, for the purpose of numerical calculation we must transform the limiting distribution as in case (ii) $\alpha = 1/2$:

$$\mathrm{P}\left(\frac{\sum_{n=1}^{\infty} \frac{c^2}{n^4\pi^4} Z_n^2}{1 + \sum_{n=1}^{\infty} \frac{c^2}{n^2\pi^2} Z_n^2} \leq x \right) = \mathrm{P}\left(x + c^2 \sum_{n=1}^{\infty} \left(\frac{x}{n^2\pi^2} - \frac{1}{n^4\pi^4} \right) Z_n^2 \geq 0 \right), \tag{5}$$

and in case (iii) $0 < \alpha < 1/2$:

$$\mathrm{P}\left(\frac{\sum_{n=1}^{\infty} \frac{1}{n^4\pi^4} Z_n^2}{\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} Z_n^2} \leq x \right) = \mathrm{P}\left(\sum_{n=1}^{\infty} \left(\frac{x}{n^2\pi^2} - \frac{1}{n^4\pi^4} \right) Z_n^2 \geq 0 \right). \tag{6}$$

We now derive the c.f. on the right-hand side of (5) and (6).

Theorem 3. *The c.f.* $\phi_2(\theta)$ *of* $x + c^2 \sum_{n=1}^{\infty} \left(\frac{x}{n^2\pi^2} - \frac{1}{n^4\pi^4} \right) Z_n^2$ *is given by:*

$$\phi_2(\theta) = e^{ix\theta} \left[\frac{\sinh \sqrt{-c^2\theta xi + \sqrt{-c^4\theta^2x^2 - 2ic^2\theta}} \sinh \sqrt{-c^2\theta xi - \sqrt{-c^4\theta^2x^2 - 2ic^2\theta}}}{\sqrt{-c^2\theta xi + \sqrt{-c^4\theta^2x^2 - 2ic^2}} \sqrt{-c^2\theta xi - \sqrt{-c^4\theta^2x^2 - 2ic^2\theta}}} \right]^{-1/2}.$$

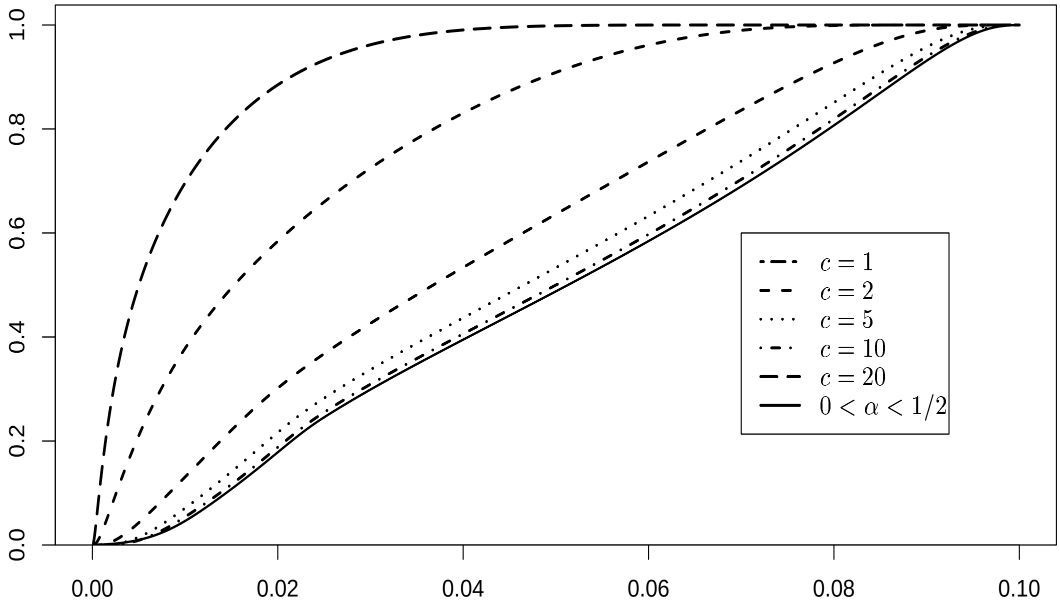

Figure 2: Distribution functions: (ii) $\alpha = 1/2$ and (iii) $0 < \alpha < 1/2$

Theorem 4. *The c.f.* $\phi_3(\theta)$ *of* $\sum_{n=1}^{\infty} \left(\frac{x}{n^2\pi^2} - \frac{1}{n^4\pi^4} \right) Z_n^2$ *is given by:*

$$\phi_3(\theta) = \left[\frac{\sinh \sqrt{-\theta xi + \sqrt{-\theta^2x^2 - 2i\theta}} \sinh \sqrt{-\theta xi - \sqrt{-\theta^2x^2 - 2i\theta}}}{\sqrt{-\theta xi + \sqrt{-\theta^2x^2 - 2i\theta}} \sqrt{-\theta xi - \sqrt{-\theta^2x^2 - 2i\theta}}} \right]^{-1/2}.$$

Figure 2 shows the asymptotic distribution functions of the normalized score statistics in cases (ii) and (iii). As described in Corollary 1, in case (ii) the asymptotic distribution reduces to that in case (iii) as c becomes large. From Figure 2, we can observe that the asymptotic distribution function shifts to the right as c becomes large and reaches the limit that corresponds to case (iii).

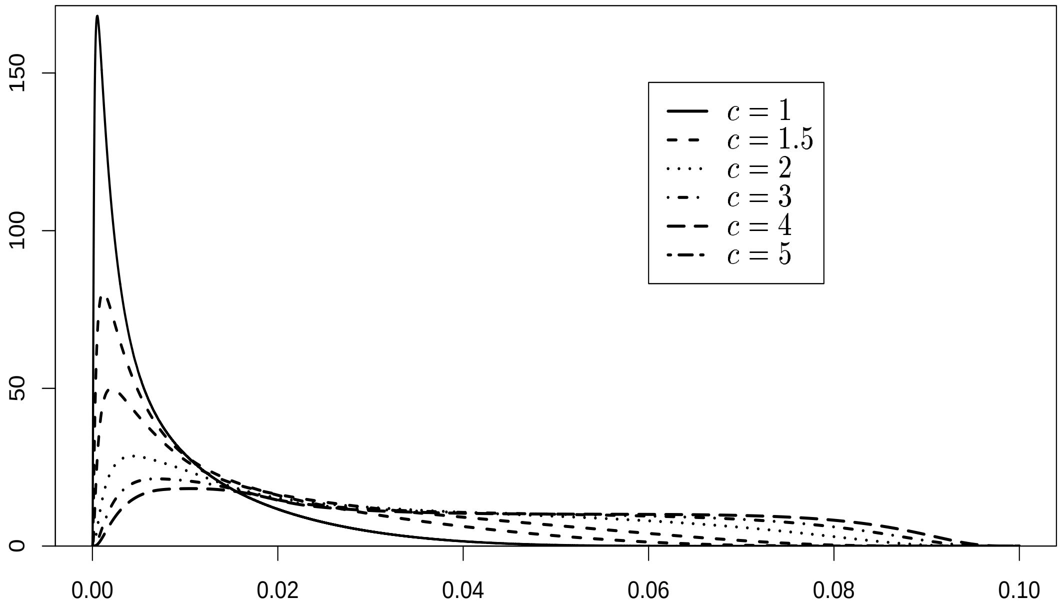

Figure 3: Density functions: (ii) $\alpha = 1/2$ for smaller c

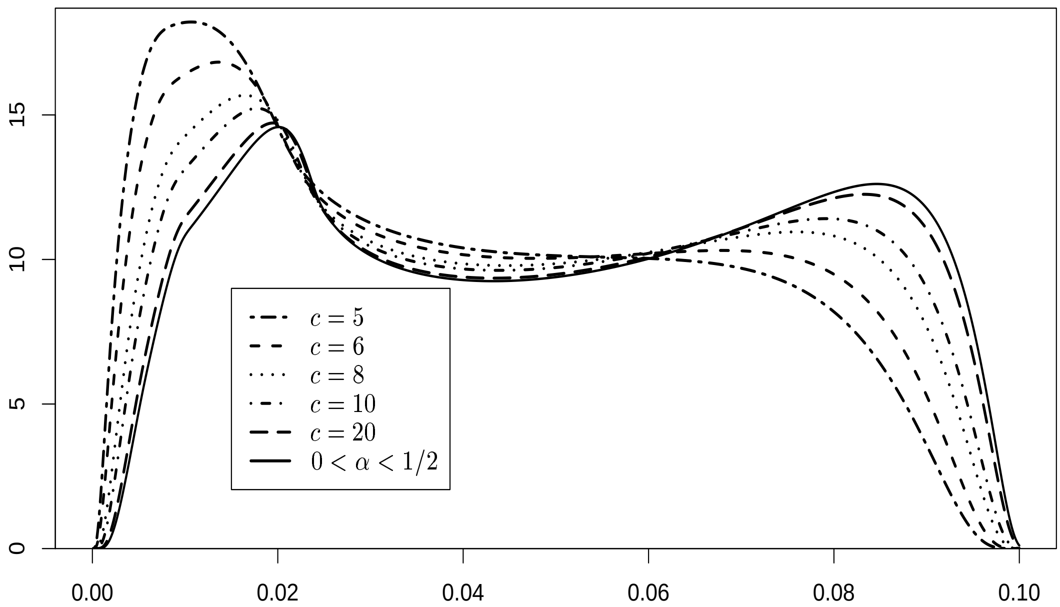

Figure 4: Density functions: (ii) $\alpha = 1/2$ for larger c and (iii) $0 < \alpha < 1/2$

Figures 3 and 4 illustrate the corresponding density functions. The former describes the density functions for a smaller c in case (ii), while the latter depicts those for a larger c in cases (ii) and (iii). In case (i), the asymptotic density function (not reported here) has a unimodal form. In contrast, the asymptotic density function in case (iii) in Figure 4 is bimodal. In the intermediate case (ii), the asymptotic density functions for a smaller c in Figure 3 and that for a larger c in Figure 4 are unimodal and bimodal, respectively. Using the unimodal and bimodal arguments, we can conclude that case (ii) has a boundary property between cases (i) and (iii).

4 EXTENSION TO A LINEAR PROCESS

In this section, the asymptotic distribution of the score statistic, as defined in (2), is derived without assuming normality. Instead of the previous N.I.D. assumption, we assume that $\{u_t\}$ is generated from the following linear process:

$$u_t = \sum_{l=0}^{\infty} \phi_l \epsilon_{t-l}, \quad \sum_{l=0}^{\infty} l|\phi_l| < \infty, \quad \sum_{l=0}^{\infty} \phi_l \neq 0, \tag{7}$$

with $\epsilon_t \sim$ i.i.d.$(0, \sigma_\epsilon^2)$. Similarly, in the N.I.D. case, we assume that $\sigma_\epsilon^2 = 1$ without loss of generality. We define the long- and short-run variances (σ_L^2, σ_S^2) by:

$$\sigma_L^2 \equiv \left(\sum_{l=0}^{\infty} \phi_l \right)^2 \quad \text{and} \quad \sigma_S^2 \equiv \sum_{l=0}^{\infty} \phi_l^2.$$

As we do not assume the normality of $\{u_t\}$, we cannot employ the eigenvalue approach. Instead, we use the weak convergence theory for the quadratic form statistic in Nabeya and Tanaka (1988) and Tanaka (1990a). These have shown the following useful results in theorem 5.

Theorem 5 (Nabeya and Tanaka (1988) and Tanaka (1990a))**.** *Let K_T be a symmetric $T \times T$ matrix and $K(s,t)$ be a continuous, symmetric and nearly positive definite function on $[0,1] \times [0,1]$. $\{u_t\}$ is defined in (7). We denote the (j,k)th element of K_T as $K_T(j,k)$. If:*

$$\max_{j,k} \left| K_T(j,k) - K\left(\frac{j}{T}, \frac{k}{T} \right) \right| \to 0 \quad as \quad T \to \infty,$$

then

$$\mathcal{L}\left(\frac{1}{T} u' K_T u \right) \to \mathcal{L}\left(\sigma_L^2 \int_0^1 \int_0^1 K(s,t) dw(s) dw(t) \right), \tag{8}$$

where $\{w(t)\}$ is a standard Brownian motion defined on $[0,1]$.

Using the above theorem, we respectively derive the asymptotic distribution of the normalized numerator and the denominator in Lemmas 3 and 4.

Lemma 3. *As $T \to \infty$ with $\rho_T = 1 - c/T^\alpha$ where $\alpha \in (0,1)$, the asymptotic distribution of the normalized denominator of the score statistic S_T is classified into three types. Each asymptotic distribution is given by:*

(i) $1/2 < \alpha < 1$

$$\frac{y'\Omega^{-1}y}{T} \to_p \sigma_S^2,$$

(ii) $\alpha = 1/2$

$$\mathcal{L}\left(\frac{y'\Omega^{-1}y}{T}\right) \to \mathcal{L}\left(\sigma_S^2 + c^2\sigma_L^2 \int_0^1 \int_0^1 K(s,t)dw(s)dw(t)\right),$$

(iii) $0 < \alpha < 1/2$

$$\mathcal{L}\left(\frac{y'\Omega^{-1}y}{T^{2-2\alpha}}\right) \to \mathcal{L}\left(c^2\sigma_L^2 \int_0^1 \int_0^1 K(s,t)dw(s)dw(t)\right),$$

where $K(s,t) = min(s,t) - st$.

Lemma 4. *As $T \to \infty$ with $\rho_T = 1 - c/T^\alpha$ where $\alpha \in (0,1)$, the asymptotic distribution of the normalized numerator of S_T is given by:*

$$\mathcal{L}\left(\frac{y'\Omega^{-2}y}{T^{4-2\alpha}}\right) \to \mathcal{L}\left(\sigma_L^2 \int_0^1 \int_0^1 K_{(2)}(s,t)dw(s)dw(t)\right),$$

where

$$K_{(2)}(s,t) = \int_0^1 K(s,u)K(u,t)du = -\frac{\min^3(s,t)}{6} - \frac{1}{2}\min(s,t)\max^2(s,t) + \frac{st(s^2+t^2)}{6} + \frac{st}{3}.$$

A combination of these results by the continuous mapping theorem yields the required asymptotic distribution given in Theorem 6.

Theorem 6. *As $T \to \infty$ with $\rho_T = 1 - c/T^\alpha$ where $\alpha \in (0,1)$, the asymptotic distribution of the normalized score statistic S_T is classified into three types. Each asymptotic distribution is given by:*

(i) $1/2 < \alpha < 1$

$$\mathcal{L}\left(\frac{S_T}{T^{3-2\alpha}}\right) \to \mathcal{L}\left(c^2\frac{\sigma_L^2}{\sigma_S^2}\frac{\int_0^1\int_0^1 K_{(2)}(s,t)dw(s)dw(t)}{\int_0^1\int_0^1 K(s,t)dw(s)dw(t)}\right),$$

(ii) $\alpha = 1/2$

$$\mathcal{L}\left(\frac{S_T}{T^2}\right) \to \mathcal{L}\left(\frac{c^2\sigma_L^2 \int_0^1\int_0^1 K_{(2)}(s,t)dw(s)dw(t)}{\sigma_S^2 + c^2\sigma_L^2 \int_0^1\int_0^1 K(s,t)dw(s)dw(t)}\right),$$

(iii) $0 < \alpha < 1/2$

$$\mathcal{L}\left(\frac{S_T}{T^2}\right) \to \mathcal{L}\left(\frac{\int_0^1\int_0^1 K_{(2)}(s,t)dw(s)dw(t)}{\int_0^1\int_0^1 K(s,t)dw(s)dw(t)}\right).$$

As in the N.I.D. case, the asymptotic distribution is classified into three types. The continuous change phenomenon of the normalized order is also observed. Moreover, the asymptotic distribution in case (iii) does not depend on the parameter c and the linear process parameters $\{\phi_l\}$.

Corollary 3. *As $T \to \infty$ followed by $c \to \infty$, the asymptotic distribution under a linear process in case (ii) $\alpha = 1/2$ converges to that in case (iii) $0 < \alpha < 1/2$ in probability.*

Corollary 4. *As $T \to \infty$ under invertibility, the asymptotic distribution of the normalized score statistic S_T under a linear process is given by:*

$$\mathcal{L}\left(\frac{S_T}{T^2}\right) \to \mathcal{L}\left(\frac{\int_0^1\int_0^1 K_{(2)}(s,t)dw(s)dw(t)}{\int_0^1\int_0^1 K(s,t)dw(s)dw(t)}\right).$$

As the proofs of Corollary 3 and Corollary 4 are similar to those of Corollary 1 and Theorem 6, respectively, they are omitted. These corollaries show that the properties of Corollary 1 and 2 are guaranteed even under a linear process. Note that by the following relations in Tanaka (1996),

$$\mathcal{L}\left(\int_0^1\int_0^1 K(s,t)dw(s)dw(t)\right) = \mathcal{L}\left(\sum_{n=1}^{\infty}\frac{Z_n^2}{n^2\pi^2}\right),$$

and

$$\mathcal{L}\left(\int_0^1\int_0^1 K_{(2)}dw(s)dw(t)\right) = \mathcal{L}\left(\sum_{n=1}^{\infty}\frac{Z_n^2}{n^4\pi^4}\right),$$

we can easily derive the limiting c.f. Given the results of the numerical calculation are similar to the N.I.D. case, we do not report them in this paper.

5 CONCLUDING REMARKS

In this paper, we derived the asymptotic distribution of the normalized Tanaka's score statistic S_T for an MA process with the largest root being moderately deviated from unity. In case (iii) $0 < \alpha < 1/2$, the score statistic has the same asymptotic distribution as that under invertibility.

In the unit root problems in an MA(1) process, derivation of the asymptotic distribution of the maximum likelihood estimator is one of the most important problems. Though there is considerable effort devoted to this problem, e.g., Sargan and Bhargava (1983), Anderson and Takemura (1986), Tanaka and Satchell (1989) and Davis and Dunsmuir (1996), it remains unproven. As we have shown, a moderate deviation process with the order parameter (iii) and an invertible process share an asymptotic property. Moreover, the asymptotic distribution in (ii) reduces to that in case (iii) as c becomes large. From these facts, we can expect that in case (iii), the asymptotic normality of the maximum likelihood estimator holds. We also anticipate that case (ii) is the boundary between noninvertibility and invertibility in an asymptotic sense. We will present such an analysis in future work.

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APPENDIX

Proof of Lemma 1. Corollary 1 in Tanaka (1990b) implies the following representation:

$$y'\Omega^{-2}y = \sum_{t=1}^{T} \frac{(1-\rho_T)^2 + \rho_T \lambda_{t,T}}{\lambda_{t,T}^2} Z_t^2, \tag{9}$$

where $\rho_T = 1 - c/T^{\alpha}$ and $\lambda_{t,T} = \sin^2(t\pi/(2(T+1)))$.

Because of (9) and the law of large numbers, it holds that:

$$
\begin{aligned}
y'\Omega^{-2}y &= \frac{c^2}{T^{2\alpha}}\sum_{t=1}^{T}\frac{1}{\lambda_{t,T}^2}Z_t^2 + \sum_{t=1}^{T}\frac{1}{\lambda_{t,T}}Z_t^2 - \frac{c}{T^{\alpha}}\sum_{t=1}^{T}\frac{1}{\lambda_{t,T}}Z_t^2 \\
&= O_p(T^{4-2\alpha}) + O_p(T^2) + O_p(T^{2-2\alpha}) = O_p(T^{4-2\alpha}).
\end{aligned}
$$

Thus, we can obtain the required result:

$$
\begin{aligned}
\frac{y'\Omega^{-2}y}{T^{4-2\alpha}} &= \frac{c^2}{T^4}\sum_{t=1}^{T}\frac{1}{\lambda_{t,T}^2}Z_t^2 + o_p(1) \\
&\Rightarrow \sum_{n=1}^{\infty}\frac{c^2}{n^4\pi^4}Z_n^2. \blacksquare
\end{aligned}
$$

Proof of Lemma 2.

Similar to the proof of Lemma 1, the denominator of S_T is similarly rewritten in the following way:

$$
\begin{aligned}
y'\Omega^{-1}y &= \sum_{t=1}^{T}\frac{(1-\rho_T)^2 + \rho_T\lambda_{t,T}}{\lambda_{t,T}}Z_t^2 \\
&= \frac{c^2}{T^{2\alpha}}\sum_{t=1}^{T}\frac{1}{\lambda_{t,T}}Z_t^2 + \sum_{t=1}^{T}Z_t^2 - \frac{c}{T^{\alpha}}\sum_{t=1}^{T}Z_t^2 \\
&= O_p(T^{2-2\alpha}) + O_p(T) + O_p(T^{1-\alpha}) \\
&= \begin{cases} O_p(T) & \text{if} \quad 1/2 < \alpha < 1 \\ O_p(T^{2-2\alpha}) + O_p(T) & \text{if} \quad \alpha = 1/2 \\ O_p(T^{2-2\alpha}) & \text{if} \quad 0 < \alpha < 1/2. \end{cases}
\end{aligned}
$$

This completes the proof.■

Proof of Theorem 1. By Lemma 1, 2 and the continuous mapping theorem, the required result is easily proved.■

Proof of Corollary 2 On the numerator of S_T:

$$
\begin{aligned}
\frac{y'\Omega^{-2}y}{T^4} &= \sum_{t=1}^{T} \frac{(1-\rho)^2 + \rho\lambda_{t,T}}{T^4\lambda_{t,T}^2} Z_t^2 \\
&= (1-\rho)^2 \sum_{t=1}^{T} \frac{Z_t^2}{T^4\lambda_{t,T}^2} + o_p(1) \\
&\Rightarrow (1-\rho)^2 \sum_{n=1}^{\infty} \frac{Z_n^2}{n^4\pi^4}.
\end{aligned}
$$

On the denominator of S_T:

$$
\begin{aligned}
\frac{y'\Omega^{-1}y}{T^2} &= \frac{1}{T^2} \sum_{t=1}^{T} \frac{(1-\rho)^2 + \rho\lambda_{t,T}}{\lambda_{t,T}} Z_t^2 \\
&= \frac{(1-\rho)^2}{T^2} \sum_{t=1}^{T} \frac{Z_t^2}{\lambda_{t,T}} + o_p(1) \\
&\Rightarrow (1-\rho)^2 \sum_{n=1}^{\infty} \frac{Z_n^2}{n^2\pi^2}.
\end{aligned}
$$

From the above arguments and the continuous mapping theorem, we obtain the required result:

$$
\frac{1}{T^2} S_T \Rightarrow \frac{\sum_{n=1}^{\infty} \frac{Z_n^2}{n^4\pi^4}}{\sum_{n=1}^{\infty} \frac{Z_n^2}{n^2\pi^2}}. \blacksquare
$$

Proof of Theorem 2 Given the sequence of $\{Z_n^2\}$ is the sequence of chi-squared random variables with one degree of freedom:

$$
\begin{aligned}
\phi_1(\theta) &= \mathrm{E}\left[\exp\left(\sum_{n=1}^{\infty} \frac{\theta c^2}{n^4\pi^4} Z_n^2\right)\right] \\
&= \prod_{n=1}^{\infty} \left(1 - \frac{2i\theta c^2}{n^4\pi^4}\right)^{-1/2} \\
&= \prod_{n=1}^{\infty} \left(1 + \frac{c\sqrt{2i\theta}}{n^2\pi^2}\right)^{-1/2} \left(1 - \frac{c\sqrt{2i\theta}}{n^2\pi^2}\right)^{-1/2} \\
&= \left(\frac{\sinh\sqrt{c\sqrt{2i\theta}}\,\sinh\sqrt{-c\sqrt{2i\theta}}}{\sqrt{c\sqrt{2i\theta}}\sqrt{-c\sqrt{2i\theta}}}\right)^{-1/2}. \blacksquare
\end{aligned}
$$

Proof of Theorem 3

$$
\begin{aligned}
\phi_2(\theta) &= e^{ix\theta} \prod_{n=1}^{\infty} \left[1 - 2c^2\theta i \left(\frac{x}{n^2\pi^2} - \frac{1}{n^4\pi^4} \right) \right]^{-1/2} \\
&= e^{ix\theta} \prod_{n=1}^{\infty} \left\{ \left[1 + \frac{1}{n^2\pi^2} \left(-c^2\theta x i + \sqrt{-c^4\theta^2 x^2 - 2ic^2\theta} \right) \right] \right. \\
&\qquad \left. \times \left[1 + \frac{1}{n^2\pi^2} \left(-c^2\theta x i - \sqrt{-c^4\theta^2 x^2 - 2ic^2\theta} \right) \right] \right\}^{-1/2} \\
&= e^{ix\theta} \left[\frac{\sinh \sqrt{-c^2\theta x i + \sqrt{-c^4\theta^2 x^2 - 2ic^2\theta}} \sinh \sqrt{-c^2\theta x i - \sqrt{-c^4\theta^2 x^2 - 2ic^2\theta}}}{\sqrt{-c^2\theta x i + \sqrt{-c^4\theta^2 x^2 - 2ic^2}} \sqrt{-c^2\theta x i - \sqrt{-c^4\theta^2 x^2 - 2ic^2\theta}}} \right]^{-1/2} . \blacksquare
\end{aligned}
$$

Proof of Theorem 4

$$
\begin{aligned}
\phi_3(\theta) &= \prod_{n=1}^{\infty} \left[1 - 2i\theta \left(\frac{x}{n^2\pi^2} - \frac{1}{n^4\pi^4} \right) \right]^{-1/2} \\
&= \prod_{n=1}^{\infty} \left\{ \left[1 + \frac{1}{n^2\pi^2} \left(-\theta x i + \sqrt{-\theta^2 x^2 - 2i\theta} \right) \right] \right. \\
&\qquad \left. \times \left[1 + \frac{1}{n^2\pi^2} \left(-\theta x i - \sqrt{-\theta^2 x^2 - 2i\theta} \right) \right] \right\}^{-1/2} \\
&= \left[\frac{\sinh \sqrt{-\theta x i + \sqrt{-\theta^2 x^2 - 2i\theta}} \sinh \sqrt{-\theta x i - \sqrt{-\theta^2 x^2 - 2i\theta}}}{\sqrt{-\theta x i + \sqrt{-\theta^2 x^2 - 2i\theta}} \sqrt{-\theta x i - \sqrt{-\theta^2 x^2 - 2i\theta}}} \right]^{-1/2} . \blacksquare
\end{aligned}
$$

Proof of Lemma 3 Tanaka (1996) has shown the following matrix relation:

$$
\Omega^{-1} = CC' - \frac{1}{T+1} Cee'C', \tag{10}
$$

where C, called a random walk generating matrix, is defined as:

$$
C = \begin{pmatrix} 1 & & \mathbf{0} \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix} \quad : (T \times T), \tag{11}
$$

and $e = (1, \cdots, 1)$ $(1 \times (T+1))$. $B(\rho_T)$ is a $T \times (T+1)$ matrix such that:

$$B(\rho_T) = \begin{pmatrix} -\rho_T & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ \mathbf{0} & & -\rho_T & 1 \end{pmatrix}. \tag{12}$$

By using (10), (11) and (12), the denominator of the score statistic S_T can be represented as:

$$y'\Omega^{-1}y = u'A_{T+1}(\rho_T)u,$$

where $u = (u_0, u_1, \ldots, u_T)'$ and $A_{T+1} = B'(\rho_T)(CC' - Cee'C'/(T+1))B(\rho_T)$.

By the law of large numbers,

$$\frac{u'u}{T} \to_p \sigma_S^2 \tag{13}$$

holds. Set the $T \times (T+1)$ matrix D such that:

$$D = \begin{pmatrix} 1 & & \mathbf{0} & 0 \\ & \ddots & & \vdots \\ \mathbf{0} & & 1 & 0 \end{pmatrix},$$

and $B_{T+1} = D'(CC' - Cee'C'/(T+1))D$. As the (j,k)th element of B_{T+1} is:

$$\begin{cases} \min(j,k) - jk/(T+1) & \text{if } \max(j,k) \neq T+1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\lim_{T\to\infty} \max_{0 \le j,k \le T} \left| \frac{B_{T+1}(j,k)}{T} - K\left(\frac{j}{T}, \frac{k}{T}\right) \right| \to 0. \tag{14}$$

By Theorem 5 and (14):

$$\frac{1}{T^2} u'D'(CC' - Cee'C'/(T+1))Du \Rightarrow \int_0^1 \int_0^1 K(s,t)dw(s)dw(t) \quad \text{as } T \to \infty. \tag{15}$$

From the results of (13) and (15), we can obtain:

$$
\begin{aligned}
u'A_{T+1}u &= u'(B(1) + \frac{c}{T^{\alpha}}D)'(CC' - Cee'C'/(T+1))(B(1) + \frac{c}{T^{\alpha}}D)u \\
&= u'u + \frac{c}{T^{2\alpha}}u'D'(CC' - Cee'C'/(T+1))Du + o_p(T) \\
&= O_p(T) + O_p(T^{2-2\alpha}) \\
&= \begin{cases} O_p(T) & \text{if } 1/2 < \alpha < 1 \\ O_p(T) + O_p(T^{2-2\alpha}) & \text{if } \alpha = \frac{1}{2} \\ O_p(T^{2-2\alpha}) & \text{if } 0 < \alpha < 1/2 \end{cases} .
\end{aligned}
$$

This completes the proof. ■

Proof of Lemma 4

We can similarly prove in the proof of Lemma 3 after some algebra and by Theorem 5.13. in Tanaka (1996) such that:

$$
\begin{aligned}
\frac{y'\Omega^{-2}y}{T^{4-2\alpha}} &= \frac{1}{T^{4-2\alpha}}u'(B(1) + \frac{c}{T^{\alpha}}D)'\Omega^{-2}(B(1) + \frac{c}{T^{\alpha}}D)u \\
&= c^2\frac{u'D\Omega^{-2}Du}{T^4} + o_p(1) \\
&\Rightarrow c^2\sigma_L^2\int_0^1\int_0^1 K_{(2)}(s,t)dw(s)dw(t). \blacksquare
\end{aligned}
$$