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A Simple Panel Stationarity Test in the Presence of Cross-Sectional Dependence

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Abstract

This paper develops a simple test for the null hypothesis of stationarity in heterogeneous panel data with cross-sectional dependence in the form of a common factor in the disturbance. We do not estimate the common factor but mop-up its effect by employing the same method as the one proposed in Pesaran (2007) in the unit root testing context. Our test is basically the same as the KPSS test but the regression is augmented by cross-sectional average of the observations. We also develop a Lagrange multiplier (LM) test allowing for cross-sectional dependence and, under restrictive assumptions, compare our augmented KPSS test with the extended LM test under the null of stationarity, under the local alternative and under the fixed alternative, and discuss the differences between these two tests. We also extend our test to the more realistic case where the shocks are serially correlated. We use Monte Carlo simulations to examine the finite sample property of the augmented KPSS test.

JEL classification: C12, C33

Key words: Panel data; stationarity; KPSS test; cross-sectional dependence; LM test; locally best test.

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1. Introduction

Since the beginning of the 90's, the theoretical and empirical econometrics literature witnessed a formidable output on testing unit root and stationarity in panel data with large T (time dimension) and N (cross-section dimension). The main motive for applying unit root and stationarity tests to panel data is to improve the power of the tests relative to their univariate counterparts. This was supported by the ensuing applications and simulations. The early theoretical contributions are by Breitung and Meyer (1994), Choi (2001), Hadri (2000), Hadri and Larsson (2005), Im, Pesaran and Shin (2003), Levin, Lin and Chu (2002), Maddala and Wu (1999), Phillips and Moon (1999), Quah (1994) and Shin and Snell (2006). On the application side, the early contributions were the work of O'Connell (1998), Oh (1996), Papell (1997, 2002), Wu (1996) and Wu and Wu (2001), who focused on testing the existence of purchasing power parity. Culver and Papell (1997) applied panel unit root tests to the inflation rate for a subset of OECD countries. They have also been employed in testing output convergence and more recently in the analysis of business cycle synchronization, house price convergence, regional migration and household income dynamics (cf. Breitung and Pesaran (2005)). All these "first generation" panel tests are based on the incredible assumption that the cross-sectional units are independent or at least not cross-sectionally correlated. Banerjee (1999), Baltagi and Kao (2000), Baltagi (2001) provide comprehensive surveys on the first generation panel tests. However, in most empirical applications this assumption is erroneous. O'Connell (1998) was the first to show via simulation that the panel tests are considerably distorted when the independence assumption is violated, whether the null hypothesis is a unit root or stationarity. Banerjee, Marcellino and Osbat (2001, 2004) argued against the use of panel unit root tests due to this problem. Therefore, it became imperative that in applications using panel tests to account for the possibility of cross-sectional dependence. This led, recently, to a flurry of papers accounting for cross-sectional dependence of different forms or second generation panel unit root tests. The most noticeable proposals in this area are by Chang (2004), Phillips and Sul (2003), Bai and Ng (2004), Moon and Perron (2004), Choi and Chue (2007) and Pesaran (2007) for unit root panel tests. For panel stationarity tests, the only contributions so far are by Bai and Ng (2005) and Harris, Leybourne and McCabe (2005), both of which corrected for cross-sectional dependence by using the principal component analysis proposed by Bai and Ng (2004).

Choi and Chue (2007) utilize subsampling technique to tackle cross-sectional dependence. Phillips and Sul (2003), Bai and Ng (2004), Moon and Perron (2004) and Pesaran (2007) employ factor models to allow for cross-sectional correlation (cf. to de Silva, Hadri and Tremayne (2007) for the comparison of the three last tests). Pesaran (2007) considers only one factor and instead of estimating it, he augments the ADF regressions with the cross-sectional averages of lagged levels and first-differences of the individual series to account for the cross-sectional dependence generated by this one factor. Other contributions are by Maddala and Wu (1999) and Chang (2004) who exploited the flexibility of the bootstrap method to deal with the pervasive problem of cross-sectional dependence of general form. Breitung and Pesaran (2005) give an excellent survey of the first and second generation panel tests.

The transfer of testing for unit root and stationarity from univariate time series to large panel data contributed to a significant increase of the power of those tests. However, this transfer led to a number of difficulties besides the problem of cross-sectional dependence. In particular, the asymptotic theory is by far more intricate due to the presence of two indices: the time dimension and the number of cross-sections. The limit theory for this class of panel data has been developed in a seminal paper by Phillips and Moon (1999). In their paper they study *inter alia* the limit theory that allows for both sequential limits, wherein $T \to \infty$ followed by $N \to \infty$, and joint limits where $T, N \to \infty$ simultaneously. They also mention, in the same paper, the diagonal path limit theory in which the passage to infinity is done along a specific diagonal path. The drawback of sequential limits is that in certain cases, they can give asymptotic results which are misleading. The downside of diagonal path limit theory is that the assumed expansion path $(T(N), N) \to \infty$ may not provide an appropriate approximation for a given (T, N) situation. Finally, the joint limit theory requires, generally, a moment condition as well as a rate condition on the relative speed of T and N going to infinity.

In this paper, we adapt Pesaran (2007) approach to the panel stationarity test of Hadri (2000) due to its conceptual simplicity. Our test is basically the same as the Kwiatkowski et al. (1992) test (KPSS test hereafter) with the regression augmented by cross-sectional average of the observations. We show that the limiting null distribution is the same as the test suggested by Hadri (2000), which is an Lagrange multiplier (LM) test without cross-sectional dependence. We also extend Hadri's test and develop the LM test allowing for cross-sectional dependence. We compare our augmented KPSS test with the extended LM test under the null of stationarity, under the local alternative and under the fixed alternative, and discuss the difference between these two tests. We then extend our test to the case of the serially correlated shocks, and use Monte Carlo simulations to examine the finite sample properties of the augmented KPSS test.

The paper is organized as follows. Section 2 sets up the model and assumptions, and define the augmented test statistic. We also develop the LM test allowing for cross-sectional dependence. Section 3 is devoted to the comparison of our augmented KPSS test under restrictive assumptions with the extended LM test under the null of stationarity, under the local alternative and under the fixed alternative. We show that the limiting null distribution of the augmented KPSS test is the same as that of Hadri's (2000) test. We also examine whether our theoretical result is valid in finite samples via simple Monte Carlo simulations. In Section 4, we relax Assumption 1 in order to allow for serial correlation in the error term and propose a modification of the augmented KPSS test statistic to correct for the presence of this serial correlation. Once again, we examine the finite sample properties of the proposed test statistic via Monte Carlo simulations. Section 5 concludes the paper. All the proofs are presented in the Appendix.

A summary word on notation. We define $M_A = I_T - A(A'A)^{-1}A'$ for a full column rank matrix A. The symbols $\xrightarrow{p,T}$ and $\xRightarrow{T}$ signify convergence in probability and weak convergence respectively as $T \to \infty$ with N fixed, while $\xrightarrow{p,N}$ and $\xRightarrow{N}$ means convergence in probability and weak convergence respectively when $N \to \infty$. We denote sequential convergence in probability and sequential weak convergence by $\overset{p,T,N}{\longrightarrow}$ and $\overset{T,N}{\Longrightarrow}$ respectively as $T \to \infty$ followed by $N \to \infty$. We denote $[Tr]$ element of $T \times 1$ vector A by $[A]_{[Tr]}$ where $[Tr]$ is the largest integer less than Tr.

2. Model and Test Statistics

2.1. Model and assumptions

Let us consider the following model:

$$y_{it} = z_t' \delta_i + r_{it} + u_{it}, \quad r_{it} = r_{it-1} + v_{it}, \quad u_{it} = f_t \gamma_i + \varepsilon_{it} \tag{1}$$

for $i = 1, \cdots, N$ and $t = 1, \cdots, T$ where z_t is deterministic and $r_{i0} = 0$ for all i. The commonly used specification of z_t in the literature is either $z_t = z_t^{\mu} = 1$ or $z_t = z_t^{\tau} = [1, t]'$. In this paper, we consider these two cases. Accordingly, we define $\delta_i = \alpha_i$ when $z = 1$ and $\delta_i = [\alpha_i, \beta_i]'$ when $z = [1, t]'$. In model (1), $z_t'\delta_i$ is the individual effect while f_t is one dimensional unobserved common factor and ε_{it} is the individual-specific (idiosyncratic) error.

By stacking y_{it} with respect to t, model (1) can be expressed as

$$\begin{bmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{iT} \end{bmatrix} = \begin{bmatrix} z_1' \\ z_2' \\ \vdots \\ z_T' \end{bmatrix} \delta_i + \begin{bmatrix} r_{i1} \\ r_{i2} \\ \vdots \\ r_{iT} \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_T \end{bmatrix} \gamma_i + \begin{bmatrix} \varepsilon_{i1} \\ \varepsilon_{i2} \\ \vdots \\ \varepsilon_{iT} \end{bmatrix},$$

$$\begin{bmatrix} r_{i1} \\ r_{i2} \\ \vdots \\ r_{iT} \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{iT} \end{bmatrix}$$

or

$$\begin{aligned} \mathbf{y}_i &= Z\delta_i + \mathbf{r}_i + \mathbf{f}\gamma_i + \boldsymbol{\varepsilon}_i \\ &= Z\delta_i + L\mathbf{v}_i + \mathbf{f}\gamma_i + \boldsymbol{\varepsilon}_i, \end{aligned} \tag{2}$$

where $Z = [\boldsymbol{\tau}, \mathbf{d}]$ with $\boldsymbol{\tau} = [1, 1, \cdots, 1]'$ and $\mathbf{d} = [1, 2, \cdots, T]'$ being $T \times 1$ vectors. Further, we have

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_N \end{bmatrix} = \begin{bmatrix} Z & & & \\ & Z & & \\ & & \ddots & \\ & & & Z \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_N \end{bmatrix} + \begin{bmatrix} L & & & \\ & L & & \\ & & \ddots & \\ & & & L \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_N \end{bmatrix} + \begin{bmatrix} \mathbf{f}\gamma_1 \\ \mathbf{f}\gamma_2 \\ \vdots \\ \mathbf{f}\gamma_N \end{bmatrix} + \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \boldsymbol{\varepsilon}_2 \\ \vdots \\ \boldsymbol{\varepsilon}_N \end{bmatrix}$$

or

$$\begin{aligned} \mathbf{y} &= (I_N \otimes Z)\delta + \mathbf{r} + (\gamma \otimes \mathbf{f}) + \boldsymbol{\varepsilon} \qquad (3) \\ &= (I_N \otimes Z)\delta + (I_N \otimes L)\mathbf{v} + (\gamma \otimes \mathbf{f}) + \boldsymbol{\varepsilon}. \end{aligned}$$

In this section, we assume the following simple assumption:

Assumption 1 *(i) The stochastic processes $\{\varepsilon_{it}\}$, $\{f_t\}$ and $\{v_{it}\}$ are independent,*

$$\varepsilon_{it} \sim i.i.d.N(0, \sigma_\varepsilon^2), \quad f_t \sim i.i.d.N(0, \sigma_f^2), \quad v_{it} \sim i.i.d.N(0, \sigma_v^2),$$

and σ_ε^2, σ_f^2 and σ_v^2 are known.
(ii) There exist real numbers M_1, $\underline{M}$ and $\overline{M}$ such that $|\gamma_i| < M_1 < \infty$ for all i and $0 < \underline{M} < |\bar{\gamma}| < \overline{M} < \infty$ for all N, where $\bar{\gamma} = N^{-1} \sum_{i=1}^{N} \gamma_i$.

Assumption 1(i) is restrictive and not practical. The assumption of normality with homoskedasticity is required to derive the LM test and to discuss the optimal property of the tests. The variances of the innovations are assumed to be known in order to make the theoretical investigation as simple as possible. In fact, if the variances are unknown, we need to estimate them and the asymptotic property of the tests depends on those estimators. Our purpose in this section is to examine the theoretical effect of "augmentation", which is explained below, on stationarity tests. Assumption 1(i) will be relaxed in Section 4 to a more practical one. Assumption 1(ii) is concerned with the weights of the common factor f_t. This assumption implies that each individual is possibly affected by the common factor with the finite weight γ_i and that the absolute value of the average of γ_i is bounded away from 0 and above both in finite samples and in asymptotics. The latter property is important in order to eliminate the common factor effect from the regression. A similar assumption is also entertained in Pesaran (2007).

We consider a test for the null hypothesis of (trend) stationarity against the alternative of a unit root for model (1). Since all the innovations are homoskedastic, the testing problem is given by

$$H_0: \ \rho \equiv \frac{\sigma_v^2}{\sigma_\varepsilon^2} = 0 \ \forall i \quad \text{v.s.} \quad H_1: \rho \geq \ 0 \tag{4}$$

where $\rho = \sigma_v^2/\sigma_\varepsilon^2$ is a signal-to-noise ratio. Under H_0, r_{it} becomes equal to zero for all i so that y_{it} is stationary whereas some or all of the cross-sectional units have a unit root under the alternative.

2.2. A simple stationarity test

A panel stationarity test has already been proposed by Hadri (2000) and Shin and Snell (2006) for cross-sectionally independent data and we extend Hadri's test to the cross-sectionally dependent case. Hadri (2000) showed that if there is no cross-sectional dependence in a model, we can construct the LM test using the regression residuals of y_{it} on z_t in the same way as KPSS (1992) and that the limiting distribution of the standardized LM test statistic is standard normal under the null hypothesis. However, it can be shown that Hadri's (2000) test depends on nuisance parameters even asymptotically if there exits cross-sectional dependence; we then need to develop a stationarity test that takes into account cross-sectional dependence.

In order to eliminate the effect of the common factor from the test statistic, we make use of the simple method proposed by Pesaran (2007), which develops panel unit root tests with cross-sectional dependence. As in Pesaran (2007), we first take a cross-sectional average of the model:

$$\bar{y}_t = z_t'\bar{\delta} + \bar{r}_t + f_t\bar{\gamma} + \bar{\varepsilon}_t, \tag{5}$$

where $\bar{y}_t = N^{-1}\sum_{i=1}^N y_{it}$, $\bar{\delta}_t = N^{-1}\sum_{i=1}^N \delta_i$, $\bar{r}_t = N^{-1}\sum_{i=1}^N r_{it}$, $\bar{\gamma}_t = N^{-1}\sum_{i=1}^N \gamma_i$ and $\bar{\varepsilon}_t = N^{-1}\sum_{i=1}^N \varepsilon_{it}$. Since $\bar{\gamma} \neq 0$ by assumption, we can solve equation (5) with respect to f_t

as follows:

$$f_t = \frac{1}{\bar{\gamma}} \left(\bar{y}_t - z_t' \bar{\delta} - \bar{r}_t - \bar{\varepsilon}_t \right).$$

By inserting this solution of f_t into model (1) we obtain the following augmented regression model:

$$y_{it} = z_t' \tilde{\delta}_i + \tilde{\gamma}_i \bar{y}_t + \epsilon_{it}, \tag{6}$$

where $\tilde{\delta}_i = \delta_i - \tilde{\gamma}_i \bar{\delta}$, $\tilde{\gamma}_i = \gamma_i / \bar{\gamma}$ and $\epsilon_{it} = r_{it} - \tilde{\gamma}_i \bar{r}_t + \varepsilon_{it} - \tilde{\gamma}_i \bar{\varepsilon}_t$. Based on (6) we propose to regress y_{it} on z_t and $\bar{y}_t$ for each i and construct the test statistic in the same way as Hadri (2000). That is,

$$Z_A = \frac{\sqrt{N}(\overline{ST} - \xi)}{\zeta} \tag{7}$$

$$\text{where} \quad \overline{ST} = \frac{1}{N} \sum_{i=1}^{N} ST_i \quad \text{with} \quad ST_i = \frac{1}{\sigma_\varepsilon^2 T^2} \mathbf{y}_i' M_w L' L M_w \mathbf{y}_i$$

$$\text{and} \quad \begin{cases} \xi = \xi_\mu = \frac{1}{6}, \quad \zeta^2 = \zeta_\mu^2 = \frac{1}{45} & \text{when } z_t = z_t^\mu = 1, \\ \xi = \xi_\tau = \frac{1}{15}, \quad \zeta^2 = \zeta_\tau^2 = \frac{11}{6300} & \text{when } z_t = z_t^\tau = [1, t]'. \end{cases}$$

Note that ST_i can also be expressed as

$$ST_i = \frac{1}{\sigma_\varepsilon^2 T^2} \sum_{t=1}^{T} (S_{it}^w)^2 \quad \text{where} \quad S_{it}^w = \sum_{s=1}^{t} \hat{\epsilon}_{is}$$

with $\hat{\epsilon}_{it}$ obtained for each i by regressing y_{it} on $w_t = [z_t', \bar{y}_t]'$ for $t = 1, \cdots, T$.

From (7) we can see that $\overline{ST}$ is the average of the KPSS test statistic across i and Z_A is normalized so that it has the limiting distribution. We call Z_A the augmented KPSS test statistic.

2.3. An LM test for panel stationarity

Although the augmented KPSS test is easy to implement, we do not know whether it has an optimal property. Note that if $\bar{y}_t$ were deterministic, we could see that Z_A is equivalent to the LM test statistic and, because the LM test is a locally best invariant test under the assumption of normality, the augmented KPSS test would also be locally optimal. One might think that this local optimality holds because $\bar{y}_t$ converges in probability to its mean, $z_t\bar{\delta} + f_t\bar{\gamma}$, as $N \to \infty$ and f_t is independent of ε_{it}. In order to investigate the local optimality of the augmented KPSS test, we consider the LM test and compare the two tests.

Under the assumption of normality, the log-likelihood function of $\mathbf{y}$, denoted by ℓ, is expressed as

$$\ell = const - \frac{1}{2}\log|\Omega| - \frac{1}{2}\{\mathbf{y} - (I_N \otimes Z)\delta\}'^{-1}\{\mathbf{y} - (I_N \otimes Z)\delta\},$$

$$\text{where} \quad \Omega \equiv Var(\mathbf{y}) = \rho\left(\sigma_\varepsilon^2 I_N \otimes LL'\right) + A \otimes I_T \quad \text{with} \quad A = \sigma_f^2\gamma\gamma' + \sigma_\varepsilon^2 I_N.$$

The partial derivative of ℓ with respect to ρ is given by

$$\frac{\partial \ell}{\partial \rho} = const + \frac{1}{2}\{\mathbf{y} - (I_N \otimes Z)\delta\}'^{-1}\frac{\partial \Omega}{\partial \rho}\Omega^{-1}\{\mathbf{y} - (I_N \otimes Z)\delta\}. \tag{8}$$

Noting that

$$\Omega|_{H_0} = A \otimes I_T, \quad \text{and} \quad \left.\frac{\partial \Omega}{\partial \rho}\right|_{H_0} = \sigma_\varepsilon^2 I_N \otimes LL', \tag{9}$$

the maximum likelihood (ML) estimator of δ under H_0 is given by

$$\begin{aligned} \hat{\delta} &= \left[(I_N \otimes Z')\,\Omega^{-1}\big|_{H_0}(I_N \otimes Z)\right]^{-1}(I_N \otimes Z')\,\Omega^{-1}\big|_{H_0}\mathbf{y} \\ &= (A^{-1} \otimes Z'Z)^{-1}(A^{-1} \otimes Z')\mathbf{y} \\ &= [I_N \otimes (Z'Z)^{-1}Z']\mathbf{y}. \end{aligned} \tag{10}$$

Thus, the maximum likelihood estimator of δ under H_0 is the same as the OLS estimator. By evaluating (8) under the null hypothesis using (9) and (10), the LM test statistic is given by

$$\begin{aligned} \overline{LM} &= \frac{1}{NT^2}\{\mathbf{y} - (I_N \otimes Z)\hat{\delta}\}'(A^{-1} \otimes I_T)(\sigma_\varepsilon^2 I_N \otimes LL')(A^{-1} \otimes I_T)\{\mathbf{y} - (I_N \otimes Z)\hat{\delta}\} \\ &= \frac{1}{NT^2}\mathbf{y}'(\sigma_\varepsilon^2 A^{-2} \otimes M_z LL' M_z)\mathbf{y}. \end{aligned}$$

By normalizing $\overline{LM}$, the extended LM test statistic for panel stationarity is given by

$$Z_{LM} = \frac{\sqrt{N}\left(\overline{LM} - \xi\right)}{\zeta} \tag{11}$$

where ξ and ζ are the same as in Z_A. Note that $\overline{LM}$ can also be expressed as

$$\overline{LM} = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_\varepsilon^2 a_2^{ij} LM_{ij} \quad \text{with} \quad LM_{ij} = \frac{1}{T^2}\sum_{t=1}^{T} S_{it}^z S_{jt}^z \quad \text{and} \quad S_{it}^z = \sum_{s=1}^{t}\hat{u}_{is},$$

where a_2^{ij} being the (i,j) element of A^{-2} and $\hat{u}_{it}$ for each i is obtained by regressing y_{it} on z_t.

3. Theoretical Property of the Augmented KPSS and LM Tests

3.1. The limiting distributions of the test statistics

In this section we compare the augmented KPSS test with the extended LM test. Note that the LM test is known to be a locally best invariant test under Assumption 1. Because there is no one-to-one transformation between Z_A and Z_{LM}, we can see that the augmented KPSS test does not have the local optimality in finite samples. Then, our interest moves on to whether it is asymptotically locally optimal or not.

In order to investigate the asymptotic local optimality of the augmented KPSS test, we compare it with the LM test statistic under the null hypothesis, under the local alternative and under the fixed alternative. We first give the limiting distributions of the two test statistics under the null hypothesis.

Theorem 1 *Assume that Assumption 1 holds. Under H_0, as $T \to \infty$ followed by $N \to \infty$, the augmented KPSS and LM test statistics have a limiting standard normal distribution for both cases of $z_t = 1$ and $z_t = [1, t]'$,*

$$Z_A, Z_{LM} \overset{T,N}{\Longrightarrow} N(0,1).$$

Note that the rejection region of Z_A and Z_{LM} is the right hand tail as in Hadri's (2000) test.

Remark 1 *Although we derived only the sequential weak limit of the test statistics, it is not difficult to see that the sequential limit is the same as the joint limit where T and N go to infinity simultaneously if we additionally assume the rate condition $N/T \to 0$. This is because all the innovations are i.i.d. normal and the variances are known (see Shin and Snell (2006)). In particular, the assumption of known variances is helpful to establish the joint limit because it is sufficient to consider the joint limit of the numerator of the test statistic in this case. According to Phillips and Moon (1999), this rate condition indicates that the joint limit theory holds when the cross-sectional dimension N is moderate while T is relatively large. However, we will relax the assumption of normality and consider the dependent shocks in a later section. In this case, it would be difficult to establish the equivalence between the sequential and joint limits, and we will rely only on the sequential limit technique. In order to see if the sequential limit theory can successfully approximate the finite sample behavior of the test statistics, we will conduct Monte Carlo simulations in a later section.*

Theorem 1 shows that Pesaran's (2007) method works well in order to eliminate cross-sectional dependence even for testing the null hypothesis of stationarity and the augmented KPSS test is asymptotically equivalent to the LM test statistic under the null hypothesis.

We next investigate the asymptotic property of the test statistics under the local alternative, which is expressed as

$$H_1^\ell: \ \rho = \frac{c^2}{\sqrt{N}T^2} \quad \text{where } c \text{ is some constant.}$$

Note that for a single time series analysis, the local alternative is given by $\rho = c^2/T^2$. Since the sum of ST_i is normalized by $\sqrt{N}$ as in Z_A, the local alternative for panel stationarity tests becomes $\rho = c^2/(\sqrt{N}T^2)$.

Theorem 2 *Assume that Assumption 1 holds. Under H_1^ℓ, as $T \to \infty$ followed by $N \to \infty$, the augmented KPSS and LM test statistics have the same limiting distribution given by*

$$Z_A, \ Z_{LM} \overset{T,N}{\Longrightarrow} N(0,1) + \frac{c^2}{\zeta} E\left[\int_0^1 F_i^v(r)^2 dr\right]$$

where $F_i^v(r) = \int_0^r B_i^v(t)dt - \int_0^r z(t)'dt \left(\int_0^1 z(t)z(t)'dt\right)^{-1} \int_0^1 z(t)B_i^v(t)dt$ *with* $B_i^v(r)$ *are independent Brownian motions, which are independent of* $B_i^{\varepsilon}(r)$*,* $z(r) = 1$ *when* $z_t = 1$ *and* $z(r) = [1, r]'$ *when* $z_t = [1, t]'$*.*

This result implies that both the augmented KPSS and extended LM test statistics have the same asymptotic local distribution. Since the LM test is locally best invariant, we can see that the augmented KPSS test has the same asymptotic local optimality.

We finally investigate the asymptotic property of the test statistics under the fixed alternative H_1. The following theorem gives the difference of powers between the two tests when the alternative is not local but far away from $\rho = 0$.

Theorem 3 *Assume that Assumption 1 holds. Under* H_1*, as* $T \to \infty$ *followed by* $N \to \infty$*,*

$$\begin{aligned} \frac{1}{\sqrt{N}T^2} Z_A &\overset{T,N}{\Longrightarrow} \frac{\rho^2}{\zeta} E_{vi}\left[\int_0^1 G_i^v(r)^2 dr\right] \\ \frac{1}{\sqrt{N}T^2} Z_{LM} &\overset{T,N}{\Longrightarrow} \frac{\rho^2}{\zeta} E\left[\int_0^1 F_i^{v*}(r)^2 dr\right], \end{aligned}$$

where $G_i^v(r) = \int_0^r B_i^v(t)dt - \int_0^r z_2'(t)dt(\int_0^1 z_2(t)z_2'(t))^{-1} \int_0^1 z_2(t)B_i^v(t)dt$ *with* $z_2(t) = [z'(t), \underline{B}^v(t)]'$ *and* $\underline{B}^v(t)$ *is a standard Brownian motion independent of* $B_i^v(t)$ *and* E_{vi} *denotes the expectation operator with respect to* $B_i^v(r)$*.*

Note that since $G_i^v(r)$ depends on $B_i^v(r)$ and $\underline{B}^v(r)$ and they are independent, we can see that $E_{vi}[\int_0^1 G_i^v(r)^2 dr]$ still depends on $\underline{B}^v(\cdot)$ and then it is stochastic, while $E[\int_0^1 F_i^{v*}(r)^2 dr]$ is deterministic. This is an interesting result because, when the asymptotic local powers are the same for two tests, it is often the case that they also have the same limiting distribution under the fixed alternative. For panel stationarity tests, the two tests have the same local asymptotic power from Theorem 2 but the powers are different under the fixed alternative from Theorem 3. This implies that although the two tests are locally optimal, they are not equivalent in a wide range under the alternative.

3.2. Finite sample property under restrictive assumptions

In this subsection we investigate how accurately does the asymptotic theory approximate the finite sample behavior of the augmented KPSS and LM tests. We consider the following data generating process for finite sample simulations:

$$y_{it} = z_t'\delta_i + r_{it} + f_t\gamma_i + \varepsilon_{it}, \quad f_t \sim i.i.d.N(0,1), \quad \varepsilon_{it} \sim i.i.d.N(0,1)$$

$$r_{it} = r_{it-1} + v_{it}, \quad v_{it} \sim i.i.d.N(0,\rho), \quad \begin{cases} H_0: \ \rho = 0 \\ H_1: \ \rho = 0.0001, \ 0.001, 0.01 \end{cases}$$

where $\delta_i = \alpha_i$ for the constant case while $\delta_i = [\alpha_i, \beta_i]'$ for the trend case with α_i and β_i being drawn from independent $U(0, 0.02)$, γ_i are drawn from $-1+U(0,4)$ for strong cross-sectional correlation case (SCC) and from $U(0, 0.02)$ for weak cross-sectional correlation case (WCC), and α_i, β_i and γ_i are fixed throughout the iterations. We consider all the pairs of $N = 10$, 20, 30, 50, 100 and $T = 10$, 20, 30, 50, 100, 200. The level of significance is 0.05 and the number of replications is 10,000 in all experiments.

Table 1 shows the sizes of the tests. We can observe that the empirical size of the augmented KPSS test is close to the nominal one when T is equal to or greater than 50 for SCC case while it is slightly undersized for WCC case. On the other hand, the size of the LM test is close to the nominal one irrespective of N and T but it is slightly undersized for SCC case while it is slightly oversized for WCC case. Overall, the null distributions of the two tests seem to be well approximated by a standard normal distribution suggested by Theorem 1 in view of the size of the tests.

Table 2 reports the powers of the tests. For given N and T, the upper, middle and lower entries are the powers of the tests for $\rho = 0.0001$, 0.001 and 0.01, respectively. From the table the powers of the tests become higher for larger ρ and T, although the tests have low power when T is small. We can also observe that the powers become higher for larger N. For example, the size of the augmented KPSS test for $T = 50$, SCC and constant case is relatively close to 0.05 for all the values of N while the empirical power when $\rho = 0.001$ is 0.145, 0.202, 0.254, 0.342 and 0.539 for $N = 10$, 20, 30, 50 and 100, respectively. Table 2 implies that the tests are consistent as proved by Theorem 3.

In order to see if the augmented KPSS test can be seen as the asymptotically locally best test indicated by Theorem 2, we calculated the size adjusted power of the tests. Figure 1 draws the power curves for selected cases. From the figure we observe that the power of the augmented KPSS test is almost the same as that of the LM test for the constant case. When a linear trend is included, the augmented KPSS test is as powerful as the LM test when ρ is small while the former is slightly less powerful than the latter for the trend case.

As a whole, the finite sample behavior of the augmented KPSS and LM tests is well approximated by the asymptotic theory established in the previous section when N and T are of moderate size.

4. Extension to general case

4.1. Modification of the augmented KPSS test

So far, we have investigated the theoretical property of the augmented KPSS test under restrictive assumptions. In this section we relax Assumptions 1(i) and consider a more practical situation. Because the LM test statistic will depend on nuisance parameters in a complicated way under general assumptions and it would be difficult to correct Z_{LM} so that it becomes free of nuisance parameters, we concentrate on the modification of the augmented KPSS test.

Since it is often the case that the observed process can be approximated by an autoregressive (AR) model, we do not consider the error component model (1) but an AR(p) model instead in this section:

$$y_{it} = z_t'\delta_i + f_t\gamma_i + \varepsilon_{it}, \quad \varepsilon_{it} = \phi_{i1}\varepsilon_{it-1} + \cdots + \phi_{ip}\varepsilon_{it-p} + \nu_{it}. \tag{12}$$

The lag length p may change depending on the cross-sectional units but we suppress the dependence of p on i for notational convenience.

Assumption 2 *(i) The stochastic process f_t is stationary with a finite fourth moment and the functional central limit theorem (FCLT) holds for the partial sum process of f_t. (ii) The stochastic process ν_{it} are independent of f_t and i.i.d.$(0, \sigma_{\nu i}^2)$ across i and t with a finite fourth moments.*

This assumption allows the common factor to be stationary but still assumes that it is independent of the idiosyncratic errors, which are finite order AR processes with the i.i.d. innovations. We assume Assumptions 1 (ii) and 2 in the rest of this section.

Since our interest is whether y_{it} are (trend) stationary or unit root processes, the testing problem is given by

$$H_0' : \ \phi_i(1) \neq 0 \ \ \forall i \quad \text{v.s.} \quad H_1' : \ \phi_i(1) = 0 \quad \text{for some } i,$$

where $\phi_i(L) = 1 - \phi_{i1}L - \cdots - \phi_{ip}L^p$.

In this case we need to modify the original KPSS test statistic for serial correlation as well as cross-sectional dependence. For the correction of cross-sectional dependence, we regress y_{it} on $w_t = [z_t', \bar{y}_t, \bar{y}_{t-1}, \cdots, \bar{y}_{t-p}]$ because ε_{it} are AR(p) processes and construct S_{it}^w using this regression residuals. Along the same line as (20) in the proof of Theorem 1 it is not difficult to see that the numerator of each ST_i weakly converges to

$$\frac{1}{T^2}\sum_{t=1}^{T}(S_{it}^w)^2 \overset{T}{\Longrightarrow} \sigma_i^2 \int_0^1 \left(V_i^\varepsilon(r) + O_p\left(\frac{1}{\sqrt{N}}\right)\right)^2 dr$$

where $\sigma_i^2 = \sigma_{\nu i}^2/(1-\phi_{i1}-\cdots-\phi_{ip})^2$ and $V_i^\varepsilon(r) = B_i^\varepsilon(r) - \int_0^r z(t)'dt\left(\int_0^1 z(t)z(t)'dt\right)^{-1}\int_0^1 z(t)dB_i^\varepsilon(t)$ with $B_i^\varepsilon(t)$ are independent standard Brownian motions. This suggests that we should divide the numerator of each ST_i by a consistent estimator of the long-run variance σ_i^2 in order to correct for serial correlation.

Several consistent estimators of the long-run variance for parametric model have been proposed in the literature for a univariate time series. For example, Leybourne and McCabe (1994) propose to correct the stationarity test for serial correlation by estimating the AR coefficients based on the ML method for the ARIMA model. Their method is also applied to panel data with no cross-sectional dependence by Shin and Snell (2006). However, our preliminary simulation shows that this method does not work well in finite samples and we do not use this method in this paper.

We next consider to make use of the new truncation rule proposed by Sul, Phillips and Choi (2005). Their method is originally developed for the prewhitening method, but it is also applicable to parametric model. We first estimate the AR(p) model augmented by the lags of $\bar{y}_t$ for each i by the least squares method

$$y_{it} = z_t'\hat{\delta}_i + \hat{\phi}_{i1}y_{it-1} + \cdots + \hat{\phi}_{ip}y_{it-p} + \hat{\psi}_{i0}\bar{y}_t + \cdots + \hat{\psi}_{ip}\bar{y}_{t-p} + \hat{\nu}_{it},$$

and construct the estimator of the long-run variance by

$$\hat{\sigma}^2_{iSPC} = \frac{\hat{\sigma}^2_{\nu i}}{(1-\hat{\phi}_i)^2} \quad \text{where} \quad \hat{\phi}_i = \min\left\{1 - \frac{1}{\sqrt{T}}, \sum_{j=1}^{p}\hat{\phi}_{ij}.\right\} \quad \text{and} \quad \hat{\sigma}^2_{\nu i} = \frac{1}{T}\sum_{t=1}^{T}\hat{\nu}^2_{it}.$$

We then propose to construct the test statistic (7) using

$$ST_i = \frac{1}{\hat{\sigma}^2_{iSPC}T^2}\sum_{t=1}^{T}(S^w_{it})^2.$$

We denote this test statistic as Z_A^{SPC}.

The other method we consider is the lag-augmented method proposed by Choi (1993) and Toda and Yamamoto (1995). According to these papers, we intentionally add an additional lag of y_t and estimate an AR($p+1$) model instead of an AR(p) model:

$$y_{it} = z_t'\tilde{\delta}_i + \tilde{\phi}_{i1}y_{it-1} + \cdots + \tilde{\phi}_{ip}y_{it-p} + \tilde{\phi}_{ip+1}y_{it-p-1} + \tilde{\psi}_{i0}\bar{y}_t + \cdots + \tilde{\psi}_{ip}\bar{y}_{t-p} + \tilde{\nu}_{it},$$

and construct the test statistic using

$$ST_i = \frac{1}{\hat{\sigma}^2_{iLA}T^2}\sum_{t=1}^{T}(S^w_{it})^2 \quad \text{where} \quad \hat{\sigma}^2_{iLA} = \frac{\hat{\sigma}^2_{\nu i}}{(1-\tilde{\phi}_{i1}-\cdots-\tilde{\phi}_{ip})^2}.$$

We denote this test statistic as Z_A^{LA}.

The consistency of $\hat{\sigma}^2_{iSPC}$ and $\hat{\sigma}^2_{iLA}$ under the null hypothesis is established in the standard way and we omit here the details. On the other hand, they are shown to diverge to infinity at a rate of T under the alternative, so that ST_i can be seen as a consistent stationarity test for univariate time series. It is also shown by using the sequential limit that the null distributions of Z_A^{SPC} and Z_A^{LA} are asymptotically standard normal in the same way as Theorem 1 while they diverge to infinity under the fixed alternative. Unfortunately, it is tedious to derive the joint limit of Z_A^{SPC} or Z_A^{LA} under general assumptions and we do not pursue it. Instead, we shall conduct Monte Carlo simulations in the next section in order to see whether or not the sequential limit theory can approximate the finite sample behavior of these tests.

4.2. Finite sample property under general assumptions

In this section we conduct Monte Carlo simulations to investigate the finite sample properties of the augmented KPSS test using the long-run variance estimated by the SPC or the LA methods in order to correct for serial correlation in the innovations. The data generating process in this subsection is given as follows:

$$y_{it} = z_t'\delta_i + f_t\gamma_i + \varepsilon_{it}, \quad \varepsilon_{it} = \phi_i\varepsilon_{it-1} + \nu_{it},$$

where $f_t \sim i.i.d.(0,1)$, $\nu_{it} \sim i.i.d.N(0,1)$, f_t and ν_{it} are independent of each other, α_i, β_i and γ_i are set as in Subsection 3.2., the ϕ_i are drawn from $0.1 + U(0, 0.8)$ under the null hypothesis and they remain fixed throughout the iterations. On the other hand, the ϕ_i are set to be equal to 1 for all i under the alternative. For the purpose of comparison, we also calculate the test statistic proposed by Harris, Leybourne and McCabe (2005) (HLM hereafter). According to HLM, we first estimate the idiosyncratic errors ε_{it} by the principal component method proposed by Bai and Ng (2004) and next apply the stationarity test proposed by Harris, McCabe and Leybourne (2003) to the estimated series of $\varepsilon_{1t}, \varepsilon_{2t}, \cdots, \varepsilon_{Nt}$. HLM method requires to predetermine the order of the autocovariance and the bandwidth parameter for the kernel estimate of the long-run variance; we set these parameters as recommended in HLM (2005).

Table 2 reports the sizes of the tests. There are no entries for HLM test when $T = 10$ because the time dimension is too short to calculate their test statistic. When only a constant is included in the model, the augmented KPSS test corrected by the SPC method tends to be undersized for moderate size of T for SCC (strong cross-correlation) case while it is oversized for small or large size of T, although the over-rejection is not so severe when $N = 100$ and $T = 200$. For WCC (weak cross-correlation) case Z_A^{SPC} is undersized except for the case of $T = 10$. The augmented KPSS test corrected by the LA method has a similar property as Z_A^{SPC} for SCC case while the size of the test is relatively well controlled for WCC case. On the other hand, the size of HLM test seems to be better controlled for moderate or large size of T, although the test becomes undersized for large size of N and small or moderate size of T.

When both a constant and a linear trend are included in the model, the overall property of Z_A^{SPC} and Z_A^{LA} is preserved while HLM test tends to be undersized for N larger than 20.

Table 4 shows the nominal powers of the tests. Because of the size distortion of the tests it is not easy to compare the powers of these tests but we observe that all the tests are less powerful for the moderate size of T due to the undersize property of the tests. In some cases the augmented KPSS test apparently dominates HLM test but the reversed relation is observed in other cases. For example, the empirical sizes of Z_A^{SPC}, Z_A^{LA} and HLM test are 0.009, 0.022 and 0.078 when $N = 10$ and $T = 30$ for the constant case with SCC, while the powers of these tests are 0.437, 0.262 and 0.218. On the other hand, the sizes of these tests are 0.058, 0.076 and 0.054 when $N = 10$ and $T = 100$ for the constant case with WCC while the powers are 0.878, 0.812 and 1.00.

Although our simulations are limited, it is difficult to recommend one of these tests because none of them dominates the others. It seems that HLM test tends to work relatively well in the constant case because the size of the test is more or less controlled in many cases and it has moderate power, whereas the augmented KPSS test with SPC correction seems to perform best in many cases corresponding to the trend case (all the other tests tend to be undersized in this case) and is most powerful in many cases.

5. Conclusion

In this paper we extended Hadri's (2000) test to correct for cross-sectional dependence à la Pesaran (2007). We showed that the limiting null distribution of this augmented KPSS test is the same as the original Hadri's test that is the LM test without cross-sectional dependence. We also extended Hadri's test by developing an LM test correcting for cross-sectional dependence. Then, we compared our augmented KPSS test with the extended LM test. We found that the augmented KPSS test is asymptotically locally optimal but it is not asymptotically equivalent to the LM test under the fixed alternative. The Monte Carlo simulations indicated that we should carefully use the panel stationarity tests because they are undersized in some cases but suffer from over rejection in other cases.

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Appendix

In this appendix, we prove the theorems only for the case where $z_t = [1, t]$. The proof for the level case with $z_t = 1$ proceeds in exactly the same way and thus we omit it.

Proof of Theorem 1

We first express $\bar{y}_t$ in matrix form. Since

$$\bar{y}_t = z_t'\bar{\delta} + \bar{r}_t + f_t\bar{\gamma} + \bar{\varepsilon}_t,$$

we have

$$\begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_T \end{bmatrix} = \begin{bmatrix} z_1' \\ z_2' \\ \vdots \\ z_T' \end{bmatrix} \bar{\delta} + \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \vdots \\ \bar{r}_T \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_T \end{bmatrix} \bar{\gamma} + \begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \vdots \\ \bar{\varepsilon}_T \end{bmatrix}$$

or

$$\bar{\mathbf{y}} = Z\bar{\delta} + \bar{\mathbf{r}} + \mathbf{f}\bar{\gamma} + \bar{\boldsymbol{\varepsilon}}. \tag{13}$$

Since $\bar{\gamma} \neq 0$, we have $\mathbf{f} = (\bar{\mathbf{y}} - Z\bar{\delta} - \bar{\mathbf{r}} - \bar{\boldsymbol{\varepsilon}})/\bar{\gamma}$. By inserting this into (2), the model becomes

$$\mathbf{y}_i = Z(\delta_i - \tilde{\gamma}_i\bar{\delta}) + \tilde{\gamma}_i\bar{\mathbf{y}} + (\mathbf{r}_i - \tilde{\gamma}_i\bar{\mathbf{r}}) + (\boldsymbol{\varepsilon}_i - \tilde{\gamma}_i\bar{\boldsymbol{\varepsilon}}) \tag{14}$$

where $\tilde{\gamma}_i = \gamma_i/\bar{\gamma}$.

Let $W = [\boldsymbol{\tau}, \mathbf{d}, \bar{\mathbf{y}}] = [Z, \bar{\mathbf{y}}]$. Under the null hypothesis, ST_i can be expressed in matrix form as

$$\begin{aligned} ST_i &= \frac{1}{\sigma_\varepsilon^2 T^2}\mathbf{y}_i' M_w L' L M_w \mathbf{y}_i \\ &= \frac{1}{\sigma_\varepsilon^2 T^2}(\boldsymbol{\varepsilon}_i - \tilde{\gamma}_i\bar{\boldsymbol{\varepsilon}})' M_w L' L M_w (\boldsymbol{\varepsilon}_i - \tilde{\gamma}_i\bar{\boldsymbol{\varepsilon}}) \end{aligned} \tag{15}$$

from (14) because $(\mathbf{r}_i - \tilde{\gamma}_i\bar{\mathbf{r}})$ disappears under H_0.

Let

$$Q = \begin{bmatrix} I_2 & -\bar{\delta} \\ 0 & 1 \end{bmatrix}, \quad D_1 = diag\{D_\tau, 1\} \quad \text{and} \quad D_2 = diag\{D_\tau, T\}$$

where $D_\tau = diag\{\sqrt{T}, T\sqrt{T}\}$ and define $W^* \equiv WQ = [Z, \bar{\mathbf{y}}^*]$ where $\bar{\mathbf{y}}^* = \bar{\mathbf{y}} - Z\bar{\delta} = \mathbf{f}\bar{\gamma} + \bar{\boldsymbol{\varepsilon}}$, so that we have the equality $M_w = M_{w^*}$ and thus we can replace M_w in (15) with M_{w^*}.

Lemma 1 *Under H_0 for $0 \leq r \leq 1$,*

$$(i) \qquad \left[\frac{1}{\sqrt{T}} L(\boldsymbol{\varepsilon}_i - \tilde{\gamma}_i \bar{\boldsymbol{\varepsilon}})\right]_{[Tr]} \overset{T}{\Longrightarrow} \sigma_\varepsilon \tilde{B}_i^\varepsilon(r) = \sigma_\varepsilon \left(B_i^\varepsilon(r) + O_p\left(\frac{1}{\sqrt{N}}\right)\right),$$

$$(ii) \qquad \left[\frac{1}{\sqrt{T}} L W^* D_1^{-1}\right]_{[Tr]} \overset{T}{\Longrightarrow} \left[\int_0^r z(t)' dt,\ \sigma_f \bar{\gamma} B^f(r) + \sigma_\varepsilon \bar{B}^\varepsilon(r)\right],$$
$$= \left[\int_0^r z(t)' dt,\ \sigma_f \bar{\gamma} B^f(r) + O_p\left(\frac{1}{\sqrt{N}}\right)\right],$$

$$(iii) \qquad D_2^{-1} W^{*\prime} W^* D_1^{-1} \overset{T}{\Longrightarrow} \begin{bmatrix} \int_0^1 z(t) z(t)' dt & \int_0^1 z(t) \left(\sigma_f \bar{\gamma} dB^f(t) + \sigma_\varepsilon d\bar{B}^\varepsilon(t)\right) \\ 0 & \bar{\gamma}^2 \sigma_f^2 + \frac{\sigma_\varepsilon^2}{N} \end{bmatrix},$$
$$= \begin{bmatrix} \int_0^1 z(t) z(t)' dt & \sigma_f \bar{\gamma} \int_0^1 z(t) dB^f(t) + O_p\left(\frac{1}{\sqrt{N}}\right) \\ 0 & \bar{\gamma}^2 \sigma_f^2 + O\left(\frac{1}{N}\right) \end{bmatrix},$$

$$(iv) \qquad D_2^{-1} W^{*\prime}(\boldsymbol{\varepsilon}_i - \tilde{\gamma}_i \bar{\boldsymbol{\varepsilon}}) \overset{T}{\Longrightarrow} \begin{bmatrix} \sigma_\varepsilon \int_0^1 z(t) d\tilde{B}_i^\varepsilon(r) \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} \sigma_\varepsilon \int_0^1 z(t) dB_i^\varepsilon(r) + O_p\left(\frac{1}{\sqrt{N}}\right) \\ 0 \end{bmatrix},$$

where $z(r) = [1, r]'$, $B_i^\varepsilon(r)$ for $i = 1 \cdots, N$ and $B^f(r)$ are independent standard Brownian motions, $\bar{B}^\varepsilon(r) = N^{-1} \sum_{i=1}^N B_i^\varepsilon(r)$ and $\tilde{B}_i^\varepsilon(r) = B_i^\varepsilon(r) - \tilde{\gamma}_i \bar{B}^\varepsilon(r)$.

Proof of Lemma 1: Weak convergences in (i) is established by the functional central limit theorem (FCLT) and the continuous mapping theorem (CMT). The second equality holds because $|\tilde{\gamma}_i| < M_1/\underline{M} < \infty$ for all i from Assumption 1(ii) and $\bar{B}^\varepsilon(r) = O_p(1/\sqrt{N})$ by the weak low of large numbers (WLLN).

(ii) Since $\bar{\mathbf{y}}^* = \mathbf{f}\bar{\gamma} + \bar{\boldsymbol{\varepsilon}}$ under H_0, we have, by the FCLT,

$$\left[\frac{1}{\sqrt{T}} L W^* D_1^{-1}\right]_{[Tr]} = \left[\frac{1}{\sqrt{T}} \sum_{t=1}^{[Tr]} z_t' D_\tau^{-1}, \frac{1}{\sqrt{T}} \sum_{t=1}^{[Tr]} \left(f_t \bar{\gamma} + \frac{1}{N} \sum_{i=1}^N \varepsilon_{it}\right)\right]$$
$$\overset{T}{\Longrightarrow} \left[\int_0^r z(t)' dt,\ \sigma_f \bar{\gamma} B^f(r) + \sigma_\varepsilon \bar{B}^\varepsilon(r)\right]. \tag{16}$$

(iii) The result of the $(1,1)$ block is easily obtained. For the $(1,2)$ block, , we can see by the FCLT that

$$\frac{1}{\sqrt{T}}\boldsymbol{\tau}'\bar{\mathbf{y}}^* = \frac{\bar{\gamma}}{\sqrt{T}}\sum_{t=1}^{T} f_t + \frac{1}{N\sqrt{T}}\sum_{t=1}^{T}\sum_{i=1}^{N}\varepsilon_{it} \overset{T}{\Longrightarrow} \sigma_f\bar{\gamma}B^f(1) + \sigma_\varepsilon\bar{B}(1), \tag{17}$$

$$\frac{1}{T\sqrt{T}}\mathbf{d}'\bar{\mathbf{y}}^* = \frac{\bar{\gamma}}{T\sqrt{T}}\sum_{t=1}^{T} t f_t + \frac{1}{NT\sqrt{T}}\sum_{t=1}^{T} t\sum_{i=1}^{N}\varepsilon_{it} \overset{T}{\Longrightarrow} \int_0^1 t\left(\sigma_f\bar{\gamma}dB^f(t) + \sigma_\varepsilon d\bar{B}(t)\right) \tag{18}$$

while for the $(3,3)$ element,

$$\begin{aligned}\frac{1}{T}\bar{\mathbf{y}}^{*\prime}\bar{\mathbf{y}}^* &= \frac{\bar{\gamma}^2}{T}\sum_{t=1}^{T} f_t^2 + \frac{1}{N^2T}\sum_{t=1}^{T}\left(\sum_{i=1}^{N}\varepsilon_{it}\right)^2 + \frac{2\bar{\gamma}}{NT}\sum_{t=1}^{T} f_t\sum_{i=1}^{N}\varepsilon_{it} \\ &\overset{p,T}{\longrightarrow} \bar{\gamma}^2\sigma_f^2 + \frac{\sigma_\varepsilon^2}{N},\end{aligned}$$

by the WLLN.

(iv) The result of the first two element is obtained by the FCLT. For the last element, we have, by the WLLN,

$$\begin{aligned}\frac{1}{T}\bar{\mathbf{y}}^{*\prime}\boldsymbol{\varepsilon}_i &= \frac{\bar{\gamma}}{T}\sum_{t=1}^{T} f_t\varepsilon_{it} + \frac{1}{NT}\sum_{t=1}^{T}\varepsilon_{it}^2 + \frac{1}{NT}\sum_{t=1}^{T}\varepsilon_{it}\sum_{j\neq i}\varepsilon_{jt} \\ &\overset{p,T}{\longrightarrow} \frac{\sigma_\varepsilon^2}{N},\end{aligned}$$

and similarly,

$$\frac{1}{T}\bar{\mathbf{y}}^{*\prime}\bar{\boldsymbol{\varepsilon}} \overset{p,T}{\longrightarrow} \frac{1}{N}\sum_{i=1}^{N}\frac{\sigma_\varepsilon^2}{N} = \frac{\sigma_\varepsilon^2}{N}.$$

We then have (iv).□

Since

$$\begin{aligned}\frac{1}{\sqrt{T}}LM_w(\boldsymbol{\varepsilon}_i - \tilde{\gamma}_i\bar{\boldsymbol{\varepsilon}}) &= \frac{1}{\sqrt{T}}L(\boldsymbol{\varepsilon}_i - \tilde{\gamma}_i\bar{\boldsymbol{\varepsilon}}) \\ &\quad -\frac{1}{\sqrt{T}}LW^*D_1^{-1}\left(D_2^{-1}W^{*\prime}W^*D_1^{-1}\right)^{-1}D_2^{-1}W^{*\prime}(\boldsymbol{\varepsilon}_i - \tilde{\gamma}_i\bar{\boldsymbol{\varepsilon}}),\end{aligned}$$

we have, by Lemma 1 and the CMT,

$$
\begin{aligned}
&\left[\frac{1}{\sqrt{T}}LM_w(\boldsymbol{\varepsilon}_i-\tilde{\gamma}_i\bar{\boldsymbol{\varepsilon}})\right]_{[Tr]} \\
&\overset{T}{\Longrightarrow} \quad \sigma_\varepsilon\left\{B_i^\varepsilon(r)-\int_0^r z(t)'dt\left(\int_0^1 z(t)z(t)'dt\right)^{-1}\int_0^1 z(t)dB_i^\varepsilon(t)+O_p\left(\frac{1}{\sqrt{N}}\right)\right\} \\
&\equiv \quad \sigma_\varepsilon V_i^\varepsilon(r)+O_p\left(\frac{1}{\sqrt{N}}\right),
\end{aligned}
\tag{19}
$$

so that

$$
ST_i \overset{T}{\Longrightarrow} \int_0^1\left(V_i^\varepsilon(r)+O_p\left(\frac{1}{\sqrt{N}}\right)\right)^2 dr. \tag{20}
$$

This implies that

$$
\overline{ST} \overset{p,T,N}{\longrightarrow} E\left[\int_0^1 V_i^\varepsilon(r)^2 dr\right]. \tag{21}
$$

By appropriately normalizing $\overline{ST}$, we obtain the result for Z_A.

In order to derive the limiting distribution of the LM test statistic, we first note that under H_0,

$$
\begin{aligned}
(I_N\otimes M_z)\mathbf{y} &= (\gamma\otimes M_z\mathbf{f})+(I_N\otimes M_z)\boldsymbol{\varepsilon} \\
&\sim N\left(0,\ (\sigma_f^2\gamma\gamma'\otimes M_z)+(\sigma_\varepsilon^2 I_N\otimes M_z)\right) \\
&= N\left(0,\ A\otimes M_z\right) \\
&= (A^{1/2}\otimes M_z)\boldsymbol{\eta}
\end{aligned}
$$

where $\boldsymbol{\eta}=[\boldsymbol{\eta}_1',\cdots,\boldsymbol{\eta}_N']'\sim N(0,I_N\otimes I_T)$. Then, $\overline{LM}$ can be expressed as

$$
\begin{aligned}
\overline{LM} &= \frac{1}{NT^2}\boldsymbol{\eta}'(A^{1/2}\otimes M_z)(\sigma_\varepsilon^2A^{-2}\otimes LL')(A^{1/2}\otimes M_z)\boldsymbol{\eta} \\
&= \frac{1}{NT^2}\boldsymbol{\eta}'(I_N\otimes M_zL)(\sigma_\varepsilon^2A^{-1}\otimes I_T)(I_N\otimes L'M_z)\boldsymbol{\eta}.
\end{aligned}
\tag{22}
$$

We first investigate the matrix A. Note that $\sigma_\varepsilon^2A^{-1}$ can be expressed as

$$
\sigma_\varepsilon^2A^{-1}=\sigma_\varepsilon^2(\sigma_f^2\gamma\gamma'+\sigma_\varepsilon^2I_N)^{-1}=\left(I_N-\frac{1}{1+\frac{\sigma_f^2}{\sigma_\varepsilon^2}\gamma'\gamma}\frac{\sigma_f^2}{\sigma_\varepsilon^2}\gamma\gamma'\right).
$$

Since $\text{rk}(\gamma\gamma') = 1$ and $(\gamma\gamma')\gamma = (\gamma'\gamma)\gamma$, the $(N-1)$ eigenvalues of $\gamma\gamma'$ are 0 and the non-zero eigenvalue is $\gamma'\gamma$, for which the corresponding eigenvector is γ. Then, there exists an $N \times N$ orthonormal matrix P such that

$$P'P = PP' = I_N \quad \text{and} \quad P'\gamma\gamma'P = \text{diag}\{\gamma'\gamma, 0, \cdots, 0\} \equiv \Lambda_\gamma.$$

This implies that

$$\begin{aligned} P'\sigma_\varepsilon^2 A^{-1} P &= I_N - \frac{1}{1 + \frac{\sigma_f^2}{\sigma_\varepsilon^2}\gamma'\gamma} \frac{\sigma_f^2}{\sigma_\varepsilon^2} \Lambda_f \\ &= \begin{bmatrix} \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_f^2 \gamma'\gamma} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \equiv \Lambda_A^{-1}. \end{aligned} \tag{23}$$

By inserting (23) into (22), we obtain

$$\begin{aligned} \overline{LM} &= \frac{1}{NT^2} \boldsymbol{\eta}'(I_N \otimes M_z L)(PP'\sigma_\varepsilon^2 A^{-1} PP' \otimes I_T)(I_N \otimes L'M_z)\boldsymbol{\eta} \\ &= \frac{1}{NT^2} \boldsymbol{\eta}^{*\prime}(I_N \otimes M_z L)(\Lambda_A^{-1} \otimes I_T)(I_N \otimes L'M_z)\boldsymbol{\eta}^* \\ &= \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_f^2\gamma'\gamma} \frac{1}{NT^2} \boldsymbol{\eta}_1^{*\prime} M_z L L' M_z \boldsymbol{\eta}_1^* + \frac{1}{NT^2} \sum_{i=2}^{N} \boldsymbol{\eta}_i^{*\prime} M_z L L' M_z \boldsymbol{\eta}_i^*, \end{aligned}$$

where $\boldsymbol{\eta}^* = [\boldsymbol{\eta}_1^{*\prime}, \cdots, \boldsymbol{\eta}_N^{*\prime}]' = (P \otimes I_T)\boldsymbol{\eta} \sim N(0, I_N \otimes I_T)$. Note that the first term converges to zero in probability as $T \to \infty$ followed by $N \to \infty$, whereas for the second term, we have, by the FCLT and the CMT,

$$\begin{aligned} &\frac{1}{T^2} \boldsymbol{\eta}_i^{*\prime} M_z L L' M_z \boldsymbol{\eta}_i^* \\ &\overset{T}{\Longrightarrow} \int_0^1 \left\{ B_i^\eta(r) - \int_0^r z(t)'dt \left(\int_0^1 z(t)z(t)'dt \right)^{-1} \int_0^1 z(t) dB_i^\eta(t) \right\}^2 dr \\ &\equiv \int_0^1 V_i^\eta(r)^2 dr \end{aligned}$$

where B_i^η for $i = 2, \cdots, N$ are independent standard Brownian motions. See also Hadri (2000). Then, by appropriately normalizing $\overline{LM}$, we obtain the result.■

Proof of Theorem 2

We first give the following lemma.

Lemma 2 *Under H_1^ℓ for $0 \le r \le 1$,*

$$(i) \quad \left[\frac{1}{\sqrt{T}} L(\mathbf{r}_i - \tilde{\gamma}_i \bar{\mathbf{r}})\right]_{[Tr]} \overset{T}{\Longrightarrow} \frac{c\sigma_\varepsilon}{N^{1/4}} \int_0^r \tilde{B}_i^v(t)dt = \frac{c\sigma_\varepsilon}{N^{1/4}} \left(\int_0^r B_i^v(t)dt + O_p\left(\frac{1}{\sqrt{N}}\right)\right),$$

$$(ii) \quad \left[\frac{1}{\sqrt{T}} LW^* D_1^{-1}\right]_{[Tr]} \overset{T}{\Longrightarrow} \left[\int_0^r z(t)'dt,\ \sigma_f \bar{\gamma} B^f(r) + \sigma_\varepsilon \bar{B}^\varepsilon(r) + \frac{c\sigma_\varepsilon}{N^{1/4}} \int_0^r \bar{B}^v(t)dt\right] = \left[\int_0^r z(t)'dt,\ \sigma_f \bar{\gamma} B^f(r) + O_p\left(\frac{1}{\sqrt{N}}\right)\right],$$

$$(iii) \quad D_2^{-1} W^{*\prime} W^* D_1^{-1} \overset{T}{\Longrightarrow} \begin{bmatrix} \int_0^1 z(t)z(t)'dt & \int_0^1 z(t)\left(\sigma_f \bar{\gamma} dB^f(t) + \sigma_\varepsilon d\bar{B}^\varepsilon(t)\right) + \frac{c\sigma_\varepsilon}{N^{1/4}} \int_0^1 z(t)\bar{B}^v(t)dt \\ 0 & \bar{\gamma}^2 \sigma_f^2 + \frac{\sigma_\varepsilon^2}{N} \end{bmatrix} = \begin{bmatrix} \int_0^1 z(t)z(t)'dt & \sigma_f \bar{\gamma} \int_0^1 z(t)dB^f(t) + O_p\left(\frac{1}{\sqrt{N}}\right) \\ 0 & \bar{\gamma}^2 \sigma_f^2 + O\left(\frac{1}{N}\right) \end{bmatrix},$$

$$(iv) \quad D_2^{-1} W^{*\prime}(\mathbf{r}_i - \tilde{\gamma}_i \bar{\mathbf{r}}) \overset{T}{\Longrightarrow} \begin{bmatrix} \frac{c\sigma_\varepsilon}{N^{1/4}} \int_0^1 z(t)\tilde{B}_i^v(t)dt \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{c\sigma_\varepsilon}{N^{1/4}} \int_0^1 z(t)B_i^v(t)dt + O_p\left(\frac{1}{\sqrt{N}}\right) \\ 0 \end{bmatrix},$$

$$(v) \quad D_2^{-1} W^{*\prime}(\boldsymbol{\varepsilon}_i - \tilde{\gamma}_i \bar{\boldsymbol{\varepsilon}}) \overset{T}{\Longrightarrow} \begin{bmatrix} \sigma_\varepsilon \int_0^1 z(t)d\tilde{B}_i^\varepsilon(r) \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_\varepsilon \int_0^1 z(t)dB_i^\varepsilon(r) + O_p\left(\frac{1}{\sqrt{N}}\right) \\ 0 \end{bmatrix}.$$

Proof of Lemma 2: (i) Since $v_{it} \sim i.i.d.(0, c^2\sigma_\varepsilon^2/(\sqrt{N}T^2))$, we have

$$\sqrt{T} r_{i[Tr]} = \sqrt{T} \sum_{t=1}^{[Tr]} v_{it} \overset{T}{\Longrightarrow} \frac{c\sigma_\varepsilon}{N^{1/4}} B_i^v(r) \tag{24}$$

by the FCLT. We then obtain the weak convergence in (i) by the CMT.

(ii) The first element has already been obtained in Theorem 1. For the second element, since $\bar{y}_t^* = \bar{r}_t + \bar{\gamma} f_t + \bar{\varepsilon}_t$ under H_1^ℓ, we have

$$\begin{aligned}
\frac{1}{\sqrt{T}} \sum_{t=1}^{[Tr]} \bar{y}_t^* &= \frac{1}{N} \sum_{i=1}^{N} \frac{1}{T} \sum_{t=1}^{[Tr]} \sqrt{T} r_{it} + \frac{\bar{\gamma}}{\sqrt{T}} \sum_{t=1}^{[Tr]} f_t + \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\sqrt{T}} \sum_{t=1}^{[Tr]} \varepsilon_{it} \\
&\overset{T}{\Longrightarrow} \frac{c\sigma_\varepsilon}{N^{1/4}} \int_0^r \bar{B}^v(t) dt + \sigma_f \bar{\gamma} B^f(r) + \sigma_\varepsilon \bar{B}^\varepsilon(r).
\end{aligned}$$

(iii) Similarly to the proof of (ii), we can see that the (1,2) block of $D_2^{-1} W^{*\prime} W D_1^{-1}$ becomes

$$\begin{aligned}
D_\tau^{-1} Z' \bar{\mathbf{y}}^* &= \begin{bmatrix} \frac{1}{\sqrt{T}} \sum_{t=1}^T (\bar{r}_t + \bar{\gamma} f_t + \bar{\varepsilon}_t) \\ \frac{1}{T\sqrt{T}} \sum_{t=1}^T t(\bar{r}_t + \bar{\gamma} f_t + \bar{\varepsilon}_t) \end{bmatrix} \\
&\overset{T}{\Longrightarrow} \begin{bmatrix} \frac{c\sigma_\varepsilon}{N^{1/4}} \bar{B}^v(1) + \sigma^f \bar{\gamma} B^f(1) + \sigma_\varepsilon \bar{B}^\varepsilon(1) \\ \frac{c\sigma_\varepsilon}{N^{1/4}} \int_0^1 t d\bar{B}^v(t) + \int_0^1 t(\sigma^f \bar{\gamma} dB^f(t) + \sigma_\varepsilon d\bar{B}^\varepsilon(t)) \end{bmatrix},
\end{aligned}$$

while the (2,2) block is given by

$$\begin{aligned}
\frac{1}{T} \bar{\mathbf{y}}^{*\prime} \bar{\mathbf{y}}^* &= \frac{1}{T} \sum_{t=1}^{T} \left(\bar{r}_t^2 + \bar{\gamma}^2 f_t^2 + \bar{\varepsilon}_t^2 + 2\bar{\gamma} \bar{r}_t f_t + 2\bar{r}_t \bar{\varepsilon}_t + 2\bar{\gamma} f_t \bar{\varepsilon}_t \right) \\
&= \frac{\bar{\gamma}^2}{T} \sum_{t=1}^{T} f_t^2 + \frac{1}{T} \sum_{t=1}^{T} \bar{\varepsilon}_t^2 + O_p\left(\frac{1}{\sqrt{T}} \right) \\
&\overset{T}{\Longrightarrow} \bar{\gamma}^2 \sigma_f^2 + \frac{\sigma_\varepsilon^2}{N}.
\end{aligned}$$

(iv) Since

$$\begin{aligned}
\frac{1}{T} \sum_{t=1}^{T} \bar{y}_t^* (r_{it} - \tilde{\gamma}_i \bar{r}_t) &= \frac{1}{T^2} \sum_{t=1}^{T} (\sqrt{T} \bar{r}_t) \sqrt{T} (r_{it} - \tilde{\gamma}_i \bar{r}_t) + \frac{1}{T\sqrt{T}} \sum_{t=1}^{T} (\bar{\gamma} f_t + \bar{\varepsilon}_t) \sqrt{T} (r_{it} - \tilde{\gamma}_i \bar{r}_t) \\
&\overset{p,T}{\longrightarrow} 0,
\end{aligned}$$

we have

$$\begin{aligned}
D_2^{-1} W^{*\prime} (\mathbf{r}_i - \tilde{\gamma}_i \bar{\mathbf{r}}) &= \begin{bmatrix} \frac{1}{\sqrt{T}} D_\tau^{-1} \sum_{t=1}^T z_t \sqrt{T} (r_{it} - \bar{r}_t) \\ \frac{1}{T} \sum_{t=1}^T \bar{y}_t^* (r_{it} - \bar{r}_t) \end{bmatrix} \\
&\overset{T}{\Longrightarrow} \begin{bmatrix} \frac{c\sigma_\varepsilon}{N^{1/4}} \int_0^1 z(t) \tilde{B}_i^v(t) dt \\ 0 \end{bmatrix}.
\end{aligned}$$

(v) The first element has already been obtained in Theorem 1. For the second element, note that

$$\frac{1}{T}\bar{\mathbf{y}}^{*\prime}(\boldsymbol{\varepsilon}_i - \tilde{\gamma}_i\bar{\boldsymbol{\varepsilon}}) = \frac{1}{T\sqrt{T}}\sum_{t=1}^{T}\sqrt{T}\bar{r}_t(\varepsilon_{it} - \tilde{\gamma}_i\bar{\varepsilon}_t) + \frac{1}{T}\sum_{t=1}^{T}(\bar{\gamma}f_t + \bar{\varepsilon}_t)(\varepsilon_{it} - \tilde{\gamma}_i\bar{\varepsilon}_t).$$

The first term on the right hand side is $O_p(1/\sqrt{T})$ while the second term converges in probability to zero as proved in Lemma 1 (iv). Then, we obtain (v).□

In order to derive the limiting distribution of Z_A under H_1^{ℓ}, note that $LM_w\mathbf{y}_i = LM_{w^*}(\boldsymbol{\varepsilon}_i - \tilde{\gamma}_i\bar{\boldsymbol{\varepsilon}}) + LM_{w^*}(\mathbf{r}_i - \tilde{\gamma}_i\bar{\mathbf{r}})$. Then, we can see from Lemma 1 (i) and Lemma 2 that

$$\left[\frac{1}{\sqrt{T}}LM_w\mathbf{y}_i\right]_{[Tr]} \overset{T}{\Longrightarrow} \left(\sigma_\varepsilon V_i^\varepsilon(r) + O_p\left(\frac{1}{\sqrt{N}}\right)\right) + \frac{c\sigma_\varepsilon}{N^{1/4}}\left(F_i^v(r) + O_p\left(\frac{1}{\sqrt{N}}\right)\right)$$

where

$$F_i^v(r) = \int_0^r B_i^v(t)dt - \int_0^r z(t)'dt\left(\int_0^1 z(t)z(t)'dt\right)^{-1}\int_0^1 z(t)B_i^v(t)dt.$$

Using this, we have

$$ST_i \overset{T}{\Longrightarrow} \int_0^1 \left\{\left(V_i^\varepsilon(r) + O_p\left(\frac{1}{\sqrt{N}}\right)\right) + \frac{c}{N^{1/4}}\left(F_i^v(r) + O_p\left(\frac{1}{\sqrt{N}}\right)\right)\right\}^2 dr.$$

Since $B_i^\varepsilon(r)$ and $B_i^v(r)$ are independent, we can see that the sum of the cross product between $V_i^\varepsilon(r)$ and $F_i^v(r)$ is $O_p(1/\sqrt{N})$, so that

$$\begin{aligned} Z_A &\overset{T}{\Longrightarrow} \frac{1}{\zeta\sqrt{N}}\sum_{i=1}^{N}\left(\int_0^1 V_i^\varepsilon(r)^2dr - \xi\right) + \frac{c^2}{\zeta N}\sum_{i=1}^{N}\int_0^1 F_i^v(r)^2dr + O_p\left(\frac{1}{\sqrt{N}}\right) \\ &\overset{N}{\Longrightarrow} N(0,1) + \frac{c^2}{\zeta}E\left[\int_0^1 F_i^v(r)^2dr\right]. \end{aligned}$$

In order to derive the limiting distribution of the LM test statistic, note that

$$(I_N \otimes M_z)\mathbf{y} = (I_N \otimes M_z)\mathbf{r} + (A^{1/2} \otimes M_z)\boldsymbol{\eta}.$$

Then, the denominator of the LM test statistic can be expressed as

$$\begin{aligned} \sqrt{N}(\overline{LM} - \xi) &= \frac{1}{\sqrt{N}T^2}\left\{\boldsymbol{\eta}'(\sigma_\varepsilon^2 A^{-2} \otimes M_zLL'M_z)\boldsymbol{\eta} - \xi\right\} \\ &\quad + \frac{1}{\sqrt{N}T^2}\mathbf{r}'(\sigma_\varepsilon^2 A^{-2} \otimes M_zLL'M_z)\mathbf{r} + \frac{2}{\sqrt{N}T^2}\boldsymbol{\eta}'(\sigma_\varepsilon^2 A^{-2} \otimes M_zLL'M_z)\boldsymbol{\varepsilon}. \end{aligned} \quad (25)$$

The first term on the right hand side of (25) converges in distribution to a standard normal distribution as $T \to \infty$ followed by $N \to \infty$ as proved in Theorem 1.

Since $A^{-2} = P\Lambda_A^{-2}P'$, the second term on the right hand side of (25) is expressed as

$$\frac{1}{\sqrt{N}T^2}\mathbf{r}'(\sigma_\varepsilon^2 A^{-2} \otimes M_z LL' M_z)\mathbf{r} = \frac{1}{\sqrt{N}T^2}\mathbf{r}'(P \otimes I_T)(\sigma_\varepsilon^2 \Lambda_A^{-2} \otimes M_z LL' M_z)(P \otimes I_T)\mathbf{r}.$$

Note that

$$(P \otimes I_T)\mathbf{r} = (P \otimes I_T)(I_N \otimes L)\mathbf{v} = (I_N \otimes L)\mathbf{v}^* = \mathbf{r}^*$$

where $\mathbf{v}^* \equiv (P \otimes I_T)\mathbf{v} \sim N(0, \rho\sigma_\varepsilon^2(I_N \otimes I_T))$ and $\mathbf{r}^* \equiv (I_N \otimes L)\mathbf{v}^*$. Since $\mathbf{v}^*$ has the same distribution as $\mathbf{v}$, we can see that $\mathbf{r}^*$ has the same distributional property as $\mathbf{r}$, so that we can apply the limit theorem used for the derivation of the limiting distribution of the augmented KPSS test statistic. That is,

$$\begin{aligned}
&\frac{1}{\sqrt{N}T^2}\mathbf{r}'(\sigma_\varepsilon^2 A^{-2} \otimes M_z LL' M_z)\mathbf{r} \\
&= \frac{1}{\sqrt{N}T^2}\mathbf{r}^{*\prime}(\sigma_\varepsilon^2 \Lambda_A^{-2} \otimes M_z LL' M_z)\mathbf{r}^* \\
&= \frac{\sigma_\varepsilon^4}{(\sigma_\varepsilon^2 + \sigma_f^2\gamma'^2}\frac{1}{\sqrt{N}T^2}\mathbf{r}_1^{*\prime} M_z LL' M_z \mathbf{r}_1^* + \frac{1}{\sqrt{N}T^2}\sum_{i=2}^{N}\mathbf{r}_i^{*\prime} M_z LL' M_z \mathbf{r}_i^* \\
&\overset{T,N}{\Longrightarrow} c^2 E\left[\int_0^1 F_i^{v*}(r)^2 dr\right],
\end{aligned}$$

where F_i^{v*} is defined as F_i^v with B_i^v replaced by B_i^{v*}, which are standard Brownian motions induced by v_{it}^*. Since v_{it}^* is independent of ε_{it}, the third term of (25) can be shown to converge in probability to 0 as $T \to \infty$ followed by $N \to \infty$. We then obtain the theorem.■

Proof of Theorem 3

We first give the following lemma, which can be proved by the FCLT and the CMT and then we omit the proof.

Lemma 3 *Under H_1^f for $0 \le r \le 1$,*

$$(i) \qquad \left[\frac{1}{T\sqrt{T}} L(\mathbf{r}_i - \tilde{\gamma}_i \bar{\mathbf{r}})\right]_{[Tr]} \overset{T}{\Longrightarrow} \rho\sigma_\varepsilon \int_0^r \tilde{B}_i^v(t)dt,$$

$$(ii) \qquad \left[\frac{1}{\sqrt{T}} L W^* D_2^{-1}\right]_{[Tr]} \overset{T}{\Longrightarrow} \left[\int_0^r z(t)'dt,\ \rho\sigma_\varepsilon \int_0^r \bar{B}^v(t)dt\right],$$

$$(iii) \qquad D_2^{-1} W^{*\prime} W^* D_2^{-1} \overset{T}{\Longrightarrow} \begin{bmatrix} \int_0^1 z(t)z(t)'dt & \rho\sigma_\varepsilon \int_0^1 z(t)\bar{B}^v(t)dt \\ \rho\sigma_\varepsilon \int_0^1 z(t)'\bar{B}^v(t)dt & \rho^2\sigma_\varepsilon^2 \int_0^1 \bar{B}^v(t)^2 dt \end{bmatrix},$$

$$(iv) \qquad \frac{1}{T} D_2^{-1} W^{*\prime}(\mathbf{r}_i - \tilde{\gamma}_i \bar{\mathbf{r}}) \overset{T}{\Longrightarrow} \begin{bmatrix} \rho\sigma_\varepsilon \int_0^1 z(t)\tilde{B}_i^v(t)dt \\ \rho^2\sigma_\varepsilon^2 \int_0^1 \bar{B}^v(t)\tilde{B}_i^v(t)dt \end{bmatrix}.$$

It is not difficult to see that $[LM_{w^*}(\mathbf{r}_i - \tilde{\gamma}_i \bar{\mathbf{r}})]_{[Tr]}$ dominates $[LM_{w^*}(\boldsymbol{\varepsilon}_i - \tilde{\gamma}_i \bar{\boldsymbol{\varepsilon}})]_{[Tr]}$ and then,

$$\frac{1}{T^2} ST_i = \frac{1}{\sigma_\varepsilon^2 T^4}(\mathbf{r}_i - \tilde{\gamma}_i \bar{\mathbf{r}}) M_{w^*} L' L M_{w^*}(\mathbf{r}_i - \tilde{\gamma}_i \bar{\mathbf{r}}) + o_p(1).$$

Since $LM_{w^*}(\mathbf{r}_i - \tilde{\gamma}_i \bar{\mathbf{r}}) = L(\mathbf{r}_i - \tilde{\gamma}_i \bar{\mathbf{r}}) - W^* D_2^{-1}(D_2^{-1} W^{*\prime} W^* D_2^{-1})^{-1} D_2^{-1}(\mathbf{r}_i - \tilde{\gamma}_i \bar{\mathbf{r}})$, we have, using Lemma 3,

$$\begin{aligned} \frac{1}{T^2} ST_i \overset{T}{\Longrightarrow}\ & \rho^2 \int_0^1 \Bigg\{ \int_0^r \tilde{B}_i^v(t)dt - \left[\int_0^r z(t)'dt,\ \int_0^r \sqrt{N}\bar{B}^v(t)dt\right] \\ & \begin{bmatrix} \int_0^1 z(t)z(t)'dt & \int_0^1 z(t)\sqrt{N}\bar{B}^v(t)dt \\ \int_0^1 z(t)'\sqrt{N}\bar{B}^v(t)dt & \int_0^1 N\bar{B}^v(t)^2 dt \end{bmatrix}^{-1} \begin{bmatrix} \int_0^1 z(t)\tilde{B}_i^v(t)dt \\ \int_0^1 \sqrt{N}\bar{B}^v(t)\tilde{B}_i^v(t)dt \end{bmatrix} \Bigg\}^2 dr \\ =\ & \rho^2 \int_0^1 \Bigg\{ \int_0^r B_i^v(t)dt \\ & - \int_0^r Z_{2N}(t)'dt \left(\int_0^1 Z_{2N}(t)Z_{2N}(t)'dt\right)^{-1} \int_0^1 Z_{2N}(t)B_i^v(t)dt \Bigg\}^2 dr + O_p\left(\frac{1}{\sqrt{N}}\right), \end{aligned}$$

where $Z_{2N}(r) = [z(t)', \sqrt{N}\bar{B}^v(r)]'$. Hence, we have

$$\begin{aligned}
\frac{1}{\sqrt{N}T^2}Z_A &= \frac{1}{\zeta N}\sum_{i=1}^{N}\frac{1}{T^2}(ST_i - \xi) \\
\overset{T}{\Longrightarrow} \quad & \frac{\rho^2}{\zeta N}\sum_{i=1}^{N}\int_0^1\left(\int_0^r B_i^v(t)dt\right)^2 dr \\
& +\frac{\rho^2}{\zeta N}\sum_{i=1}^{N}\int_0^1\left\{\int_0^r Z_{2N}(t)'dt\left(\int_0^1 Z_{2N}(t)Z_{2N}(t)'dt\right)^{-1}\int_0^1 Z_{2N}(t)B_i^v(t)dt\right\}^2 dr \\
& -\frac{2\rho^2}{\zeta N}\sum_{i=1}^{N}\int_0^1\int_0^r Z_{2N}(t)'dt\left(\int_0^1 Z_{2N}(t)Z_{2N}(t)'dt\right)^{-1}\int_0^1 Z_{2N}(t)B_i^v(t)dt\int_0^r B_i^v(t)dtdr \\
& +O_p\left(\frac{1}{\sqrt{N}}\right). \qquad (26)
\end{aligned}$$

The first term of (26) converges in probability to $(\rho^2/\zeta)E[\int_0^1(\int_0^r B_i^v(t)dt)^2dr]$ by the WLLN.

For the second term, we note that

$$\frac{1}{N}\sum_{i=1}^{N}\int_0^1\int_0^1\left((B_i^v(s)B_i^v(t) - \min(s,t)\right)Z_{2N}(s)Z_{2N}(t)'dsdt \overset{p,N}{\longrightarrow} 0 \qquad (27)$$

because

$$\begin{aligned}
& E\left[\left\|\frac{1}{N}\sum_{i=1}^{N}((B_i^v(s)B_i^v(t) - \min(s,t))Z_{2N}(s)Z_{2N}(t)\prime\right\|\right] \\
\leq \quad & \left\{E\left[\left\|Z_{2N}(s)Z_{2N}(t)'\right\|^2\right]E\left[\frac{1}{N}\sum_{i=1}^{N}(B_i^v(s)B_i^v(t) - \min(s,t))^2\right]\right\}^{1/2} \leq \frac{C}{\sqrt{N}}
\end{aligned}$$

where C is some constant. Since $\sqrt{N}\bar{B}^v(t) \overset{N}{\Longrightarrow} \underline{B}^v(t)$, which is a standard Brownian motion independent of $B_i^v(t)$, we have

$$Z_{2N}(r) \overset{N}{\Longrightarrow} [z(r)', B_2^v(r)]' \equiv z_2(r). \qquad (28)$$

Hence, we observe from (27) and (28) that the second term of (26) weakly converges to

$$\begin{aligned}
\frac{\rho^2}{\zeta}H_1 \quad \equiv \quad & \frac{\rho^2}{\zeta}\int_0^1\left\{\int_0^r z_2(t)'dt\left(\int_0^1 z_2(t)z_2(t)'dt\right)^{-1}\right. \\
& \left.\int_0^1\int_0^1 \min(s,t)z_2(s)z_2(t)'dsdt\left(\int_0^1 z_2(t)z_2(t)'dt\right)^{-1}\int_0^r z_2(t)dt\right\}dr \quad (29)
\end{aligned}$$

as $N \to \infty$.

In exactly the same manner, we can see that as $N \to \infty$ the third term of (26) weakly converges to

$$\frac{\rho^2}{\zeta} H_2 \equiv \frac{\rho^2}{\zeta} \int_0^1 \left\{ \int_0^r z_2(t)' dt \left(\int_0^1 z_2(t) z_2(t)' dt \right)^{-1} \int_0^1 \int_0^r \min(s,t) z_2(t) ds dt \right\} dr. \quad (30)$$

From (26), (29) and (30), we conclude that

$$\frac{1}{\sqrt{N} T^2} Z_A \overset{T,N}{\Longrightarrow} \frac{\rho^2}{\zeta} \left\{ E \left[\int_0^1 \left(\int_0^r B_i^v(t) dt \right)^2 dr \right] + H_1 - 2H_2 \right\}.$$

Since the term in braces is the same as $E_v[\int_0^1 G_i^v(r)^2 dr]$, we have the result for Z_A.

The limiting distribution of the LM test statistic can be derived in the same way using the FCLT and the CMT. ■

Table 1. Size of the tests: base case

		constant case				trend case			
		SCC		WCC		SCC		WCC	
N	T	Z_A	Z_{LM}	Z_A	Z_{LM}	Z_A	Z_{LM}	Z_A	Z_{LM}
10	10	0.023	0.038	0.019	0.066	0.002	0.022	0.007	0.055
	20	0.046	0.035	0.026	0.064	0.026	0.026	0.014	0.061
	30	0.035	0.034	0.028	0.065	0.032	0.025	0.017	0.063
	50	0.049	0.033	0.026	0.061	0.040	0.022	0.018	0.062
	100	0.053	0.034	0.033	0.067	0.045	0.026	0.024	0.063
	200	0.057	0.036	0.036	0.066	0.048	0.026	0.026	0.063
20	10	0.012	0.038	0.014	0.057	0.000	0.026	0.005	0.045
	20	0.040	0.037	0.022	0.060	0.016	0.031	0.013	0.058
	30	0.027	0.042	0.031	0.065	0.026	0.030	0.017	0.059
	50	0.052	0.041	0.032	0.065	0.040	0.029	0.023	0.060
	100	0.057	0.042	0.040	0.067	0.043	0.030	0.023	0.058
	200	0.058	0.045	0.041	0.067	0.047	0.031	0.029	0.060
30	10	0.007	0.037	0.010	0.054	0.000	0.024	0.002	0.040
	20	0.042	0.042	0.024	0.063	0.015	0.032	0.012	0.054
	30	0.023	0.040	0.027	0.059	0.027	0.033	0.017	0.059
	50	0.053	0.041	0.034	0.060	0.041	0.031	0.021	0.056
	100	0.056	0.043	0.037	0.060	0.047	0.032	0.027	0.057
	200	0.054	0.040	0.037	0.059	0.046	0.032	0.029	0.057
50	10	0.004	0.042	0.011	0.057	0.000	0.020	0.001	0.032
	20	0.037	0.042	0.021	0.055	0.009	0.030	0.010	0.046
	30	0.015	0.042	0.024	0.057	0.018	0.033	0.015	0.052
	50	0.051	0.046	0.034	0.060	0.041	0.036	0.022	0.058
	100	0.055	0.044	0.036	0.058	0.049	0.039	0.032	0.061
	200	0.056	0.046	0.042	0.063	0.048	0.036	0.033	0.055
100	10	0.001	0.040	0.006	0.051	0.000	0.014	0.000	0.019
	20	0.037	0.045	0.015	0.055	0.006	0.031	0.006	0.043
	30	0.013	0.046	0.026	0.058	0.020	0.037	0.012	0.050
	50	0.061	0.047	0.030	0.058	0.046	0.038	0.019	0.052
	100	0.064	0.046	0.036	0.056	0.060	0.040	0.029	0.056
	200	0.060	0.040	0.038	0.052	0.064	0.040	0.033	0.055

Table 2. Power of the tests: base case

		constant case				trend case			
		SCC		WCC		SCC		WCC	
N	T	Z_A	Z_{LM}	Z_A	Z_{LM}	Z_A	Z_{LM}	Z_A	Z_{LM}
10	10	0.022	0.034	0.019	0.067	0.002	0.020	0.006	0.052
		0.022	0.038	0.019	0.070	0.002	0.020	0.007	0.054
		0.038	0.062	0.032	0.108	0.002	0.025	0.008	0.063
	20	0.042	0.035	0.024	0.062	0.027	0.026	0.014	0.063
		0.051	0.041	0.029	0.075	0.029	0.030	0.016	0.067
		0.196	0.176	0.126	0.265	0.048	0.050	0.029	0.107
	30	0.036	0.034	0.027	0.065	0.034	0.026	0.017	0.063
		0.055	0.057	0.044	0.100	0.038	0.029	0.018	0.071
		0.357	0.421	0.344	0.546	0.103	0.088	0.058	0.179
	50	0.055	0.038	0.034	0.069	0.045	0.025	0.019	0.066
		0.145	0.109	0.092	0.181	0.061	0.037	0.030	0.088
		0.882	0.862	0.796	0.923	0.356	0.297	0.243	0.453
	100	0.087	0.059	0.055	0.106	0.049	0.028	0.025	0.070
		0.520	0.462	0.434	0.588	0.143	0.097	0.086	0.193
		1.000	1.000	0.997	1.000	0.948	0.929	0.904	0.973
	200	0.224	0.178	0.171	0.270	0.073	0.044	0.043	0.103
		0.970	0.962	0.955	0.985	0.582	0.506	0.488	0.670
		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
20	10	0.011	0.039	0.015	0.059	0.000	0.024	0.005	0.045
		0.012	0.042	0.016	0.062	0.000	0.024	0.005	0.045
		0.023	0.077	0.034	0.115	0.001	0.030	0.007	0.057
	20	0.042	0.038	0.023	0.060	0.020	0.033	0.015	0.061
		0.053	0.050	0.029	0.079	0.022	0.037	0.016	0.065
		0.296	0.289	0.194	0.368	0.044	0.073	0.035	0.127
	30	0.025	0.044	0.031	0.067	0.030	0.032	0.018	0.060
		0.048	0.079	0.055	0.117	0.034	0.039	0.023	0.072
		0.549	0.684	0.555	0.759	0.130	0.150	0.097	0.230
	50	0.062	0.049	0.039	0.077	0.042	0.032	0.025	0.063
		0.202	0.178	0.148	0.238	0.066	0.053	0.039	0.096
		0.989	0.988	0.963	0.994	0.559	0.526	0.447	0.646
	100	0.102	0.081	0.072	0.118	0.053	0.037	0.032	0.069
		0.759	0.729	0.695	0.800	0.204	0.164	0.145	0.255
		1.000	1.000	1.000	1.000	0.999	0.999	0.996	1.000
	200	0.341	0.303	0.291	0.388	0.097	0.071	0.063	0.122
		1.000	0.999	0.999	1.000	0.834	0.801	0.781	0.876
		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 2. Power of the tests: base case (continued)

		constant case				trend case			
		SCC		WCC		SCC		WCC	
N	T	Z_A	Z_{LM}	Z_A	Z_{LM}	Z_A	Z_{LM}	Z_A	Z_{LM}
30	10	0.009	0.039	0.011	0.057	0.000	0.022	0.003	0.038
		0.010	0.043	0.013	0.063	0.000	0.023	0.003	0.039
		0.025	0.096	0.033	0.128	0.000	0.030	0.005	0.050
	20	0.039	0.038	0.020	0.058	0.015	0.032	0.010	0.053
		0.054	0.054	0.027	0.079	0.017	0.036	0.011	0.058
		0.385	0.391	0.251	0.465	0.041	0.080	0.033	0.126
	30	0.023	0.044	0.029	0.064	0.025	0.030	0.014	0.054
		0.047	0.086	0.061	0.119	0.031	0.039	0.019	0.068
		0.701	0.827	0.706	0.870	0.164	0.195	0.119	0.277
	50	0.067	0.054	0.041	0.076	0.041	0.033	0.021	0.057
		0.254	0.226	0.181	0.287	0.071	0.059	0.039	0.099
		0.999	0.999	0.988	0.999	0.713	0.692	0.589	0.779
	100	0.126	0.099	0.088	0.135	0.062	0.045	0.038	0.075
		0.888	0.870	0.841	0.908	0.276	0.231	0.201	0.321
		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	200	0.434	0.392	0.376	0.462	0.122	0.087	0.079	0.140
		1.000	1.000	1.000	1.000	0.943	0.926	0.912	0.957
		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
50	10	0.004	0.045	0.011	0.058	0.000	0.020	0.002	0.029
		0.005	0.049	0.012	0.066	0.000	0.020	0.002	0.030
		0.015	0.118	0.033	0.149	0.000	0.027	0.002	0.043
	20	0.039	0.046	0.022	0.062	0.010	0.030	0.010	0.049
		0.063	0.071	0.033	0.088	0.012	0.036	0.011	0.054
		0.512	0.540	0.342	0.598	0.037	0.106	0.038	0.146
	30	0.017	0.044	0.026	0.061	0.022	0.033	0.013	0.050
		0.043	0.108	0.069	0.142	0.028	0.044	0.019	0.064
		0.859	0.951	0.849	0.963	0.203	0.292	0.170	0.362
	50	0.066	0.056	0.044	0.075	0.044	0.041	0.026	0.063
		0.342	0.318	0.253	0.373	0.084	0.081	0.051	0.114
		1.000	1.000	0.998	1.000	0.878	0.877	0.787	0.913
	100	0.141	0.121	0.105	0.151	0.066	0.054	0.044	0.081
		0.975	0.971	0.958	0.980	0.373	0.343	0.292	0.419
		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	200	0.582	0.554	0.530	0.609	0.143	0.118	0.108	0.164
		1.000	1.000	1.000	1.000	0.992	0.991	0.989	0.994
		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 2. Power of the tests: base case (continued)

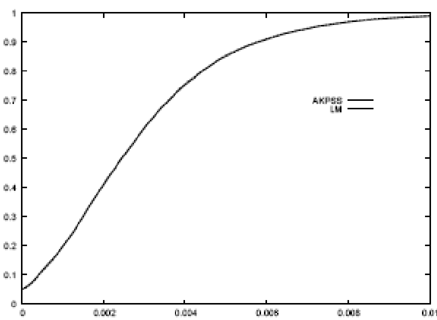
		constant case				trend case			
		SCC		WCC		SCC		WCC	
N	T	Z_A	Z_{LM}	Z_A	Z_{LM}	Z_A	Z_{LM}	Z_A	Z_{LM}
		0.001	0.038	0.006	0.048	0.000	0.014	0.000	0.020
	10	0.001	0.043	0.007	0.054	0.000	0.015	0.000	0.022
		0.010	0.151	0.032	0.174	0.000	0.025	0.001	0.035
		0.042	0.049	0.017	0.061	0.005	0.032	0.007	0.042
	20	0.076	0.087	0.033	0.103	0.006	0.036	0.008	0.049
		0.773	0.804	0.532	0.831	0.037	0.151	0.039	0.188
		0.013	0.052	0.027	0.064	0.021	0.038	0.012	0.052
	30	0.048	0.151	0.088	0.179	0.031	0.055	0.019	0.074
		0.983	0.998	0.956	0.999	0.366	0.498	0.279	0.554
		0.085	0.066	0.044	0.080	0.054	0.042	0.023	0.057
100	50	0.539	0.499	0.391	0.541	0.124	0.108	0.061	0.138
		1.000	1.000	1.000	1.000	0.991	0.991	0.960	0.994
		0.218	0.174	0.146	0.200	0.090	0.066	0.050	0.085
	100	1.000	1.000	0.999	1.000	0.629	0.573	0.501	0.626
		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		0.832	0.803	0.780	0.827	0.237	0.179	0.158	0.223
	200	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 3. Size of the tests: serially correlated case

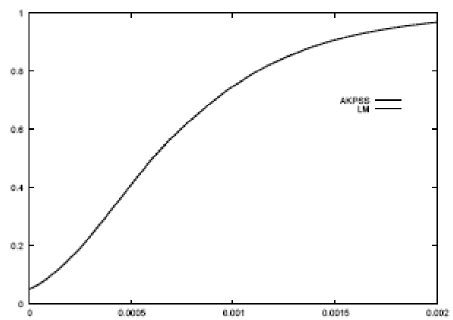
		constant case						trend case					
		SCC			WCC			SCC			WCC		
N	T	Z_A^{SPC}	Z_A^{LA}	HLM	Z_A^{SPC}	Z_A^{LA}	HLM	Z_A^{SPC}	Z_A^{LA}	HLM	Z_A^{SPC}	Z_A^{LA}	HLM
10	10	0.075	0.338	-	0.062	0.262	-	0.289	0.650	-	0.535	0.800	-
	20	0.004	0.068	0.033	0.004	0.069	0.038	0.001	0.029	0.027	0.003	0.039	0.034
	30	0.009	0.022	0.078	0.009	0.036	0.086	0.006	0.021	0.059	0.011	0.029	0.068
	50	0.040	0.062	0.086	0.018	0.046	0.079	0.030	0.050	0.056	0.014	0.034	0.056
	100	0.061	0.101	0.064	0.024	0.070	0.064	0.045	0.085	0.033	0.014	0.060	0.033
	200	0.109	0.124	0.058	0.058	0.076	0.054	0.120	0.135	0.051	0.053	0.073	0.053
20	10	0.081	0.425	-	0.080	0.338	-	0.437	0.859	-	0.759	0.922	-
	20	0.002	0.059	0.011	0.002	0.067	0.015	0.001	0.021	0.010	0.001	0.036	0.013
	30	0.002	0.008	0.043	0.004	0.033	0.058	0.001	0.008	0.023	0.006	0.022	0.031
	50	0.025	0.048	0.088	0.009	0.042	0.075	0.013	0.026	0.037	0.006	0.025	0.031
	100	0.041	0.087	0.074	0.013	0.072	0.072	0.026	0.059	0.024	0.006	0.044	0.022
	200	0.122	0.150	0.055	0.040	0.071	0.055	0.121	0.154	0.036	0.029	0.063	0.038
30	10	0.088	0.499	-	0.085	0.386	-	0.488	0.930	-	0.832	0.948	-
	20	0.001	0.068	0.003	0.001	0.063	0.006	0.001	0.020	0.003	0.002	0.039	0.007
	30	0.001	0.005	0.026	0.003	0.029	0.040	0.001	0.006	0.009	0.004	0.016	0.016
	50	0.020	0.045	0.076	0.007	0.041	0.064	0.010	0.027	0.019	0.006	0.023	0.014
	100	0.034	0.078	0.062	0.013	0.063	0.063	0.017	0.050	0.014	0.004	0.036	0.014
	200	0.131	0.176	0.058	0.027	0.071	0.059	0.145	0.187	0.028	0.020	0.059	0.031
50	10	0.089	0.635	-	0.103	0.444	-	0.603	0.984	-	0.917	0.980	-
	20	0.001	0.045	0.000	0.002	0.057	0.002	0.000	0.017	0.001	0.004	0.042	0.003
	30	0.000	0.001	0.013	0.001	0.029	0.020	0.000	0.002	0.003	0.002	0.012	0.005
	50	0.009	0.032	0.050	0.002	0.041	0.042	0.002	0.013	0.005	0.002	0.016	0.004
	100	0.030	0.076	0.059	0.006	0.049	0.060	0.017	0.040	0.010	0.004	0.029	0.009
	200	0.089	0.122	0.056	0.023	0.061	0.058	0.082	0.118	0.025	0.016	0.045	0.026
100	10	0.097	0.752	-	0.163	0.529	-	0.877	1.000	-	0.986	0.996	-
	20	0.000	0.040	0.000	0.001	0.054	0.000	0.000	0.014	0.000	0.007	0.059	0.000
	30	0.000	0.001	0.001	0.001	0.035	0.002	0.000	0.001	0.000	0.001	0.008	0.000
	50	0.003	0.018	0.018	0.001	0.032	0.013	0.001	0.003	0.000	0.001	0.007	0.000
	100	0.028	0.067	0.045	0.005	0.050	0.049	0.015	0.030	0.002	0.001	0.019	0.002
	200	0.084	0.124	0.049	0.016	0.049	0.056	0.078	0.114	0.013	0.009	0.032	0.014

Table 4. Power of the tests: serially correlated case

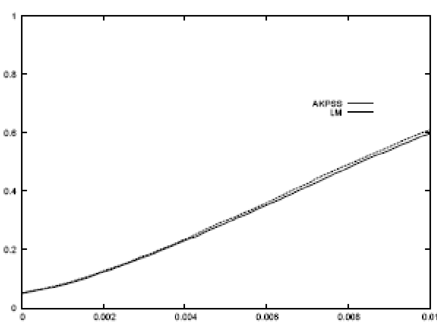
		constant case						trend case					
		SCC			WCC			SCC			WCC		
N	T	Z_A^{SPC}	Z_A^{LA}	HLM	Z_A^{SPC}	Z_A^{LA}	HLM	Z_A^{SPC}	Z_A^{LA}	HLM	Z_A^{SPC}	Z_A^{LA}	HLM
10	10	0.229	0.564	-	0.092	0.315	-	0.186	0.537	-	0.373	0.726	-
	20	0.323	0.282	0.033	0.162	0.269	0.040	0.000	0.044	0.004	0.000	0.070	0.006
	30	0.437	0.262	0.218	0.267	0.324	0.231	0.003	0.041	0.001	0.002	0.079	0.001
	50	0.695	0.373	0.740	0.454	0.461	0.739	0.039	0.086	0.000	0.023	0.155	0.001
	100	0.843	0.521	0.985	0.669	0.631	0.984	0.374	0.207	0.113	0.260	0.357	0.113
	200	0.944	0.672	1.000	0.878	0.812	1.000	0.831	0.413	0.890	0.700	0.636	0.894
20	10	0.312	0.748	-	0.123	0.392	-	0.222	0.705	-	0.508	0.857	-
	20	0.511	0.445	0.004	0.205	0.336	0.008	0.000	0.035	0.000	0.000	0.068	0.001
	30	0.609	0.407	0.194	0.336	0.413	0.200	0.000	0.036	0.000	0.001	0.085	0.000
	50	0.862	0.587	0.894	0.503	0.541	0.890	0.042	0.100	0.000	0.021	0.187	0.000
	100	0.944	0.748	1.000	0.714	0.722	1.000	0.606	0.297	0.083	0.344	0.464	0.084
	200	0.993	0.861	1.000	0.930	0.903	1.000	0.965	0.606	0.987	0.790	0.777	0.986
30	10	0.367	0.814	-	0.146	0.420	-	0.251	0.817	-	0.586	0.899	-
	20	0.608	0.574	0.001	0.231	0.376	0.002	0.000	0.030	0.000	0.000	0.064	0.000
	30	0.659	0.512	0.151	0.369	0.450	0.160	0.000	0.029	0.000	0.000	0.080	0.000
	50	0.898	0.716	0.949	0.525	0.578	0.950	0.046	0.114	0.000	0.019	0.195	0.000
	100	0.962	0.844	1.000	0.728	0.762	1.000	0.710	0.390	0.063	0.394	0.518	0.060
	200	0.996	0.921	1.000	0.947	0.930	1.000	0.981	0.765	0.998	0.831	0.844	0.999
50	10	0.449	0.928	-	0.167	0.452	-	0.288	0.919	-	0.703	0.945	-
	20	0.807	0.739	0.000	0.267	0.410	0.000	0.000	0.020	0.000	0.000	0.072	0.000
	30	0.762	0.612	0.114	0.399	0.490	0.126	0.000	0.018	0.000	0.000	0.088	0.000
	50	0.977	0.872	0.989	0.546	0.610	0.989	0.051	0.107	0.000	0.019	0.222	0.000
	100	0.995	0.943	1.000	0.752	0.788	1.000	0.895	0.469	0.028	0.456	0.582	0.028
	200	1.000	0.980	1.000	0.967	0.962	1.000	0.999	0.874	1.000	0.858	0.893	1.000
100	10	0.556	0.980	-	0.197	0.490	-	0.393	0.987	-	0.837	0.978	-
	20	0.872	0.870	0.000	0.294	0.440	0.000	0.000	0.010	0.000	0.000	0.074	0.000
	30	0.816	0.739	0.055	0.423	0.526	0.065	0.000	0.012	0.000	0.000	0.092	0.000
	50	0.986	0.947	1.000	0.575	0.646	1.000	0.050	0.118	0.000	0.019	0.254	0.000
	100	0.999	0.977	1.000	0.775	0.825	1.000	0.953	0.659	0.008	0.514	0.650	0.007
	200	1.000	0.993	1.000	0.981	0.979	1.000	1.000	0.973	1.000	0.887	0.938	1.000



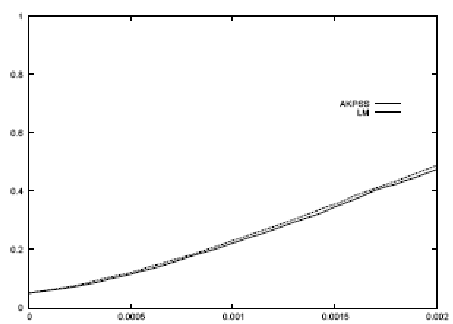
(i-a) $N = 50$, $T = 50$, constant, HCC



(i-b) $N = 50$, $T = 100$, constant, HCC



(i-c) $N = 50$, $T = 50$, trend, HCC



(i-d) $N = 50$, $T = 100$, trend, HCC

Figure 1: Finite sample power under restrictive assumptions