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How are shocks to trend and cycle correlated? A simple methodology for unidentified unobserved components models*

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Abstract

In this paper, we propose a simple methodology for investigating how shocks to trend and cycle are correlated in *unidentified* unobserved components models, in which the correlation is not identified. The proposed methodology is applied to U.S. and U.K. real GDP data. We find that the correlation parameters are negative for both countries. We also investigate how changing the identification restriction results in different trend and cycle estimates.

Keywords: Unobserved components model, Trend, Cycle, Business Cycle Analysis.

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1 Introduction

In business cycle analysis, it is often supposed that real GDP consists of two unobserved components (UC): a permanent and a transitory component. Shocks to the permanent component have long-lasting effects, whereas shocks to the transitory component are temporary and vanish in the long run. Following convention, we refer to these two components as "trend" and "cycle", respectively. Estimating these two components has been an important issue in business cycle analysis. The UC model is one of the most commonly used models for this purpose (see, for example, Watson (1986), Clark (1987), Basistha (2007), Basistha and Nelson (2007), Oh, Zivot, and Creal (2008), Sinclair (2009) and references therein for applications of UC models).

In this model, the trend is assumed to be a random walk process and the cycle is assumed to be a stationary process. It is conventional to assume that shocks to the trend and cycle are uncorrelated for the identification of model parameters although this assumption is unreasonable, as argued by Clark (1987, pp.800–801) and Zarnowitz and Ozyildirim (2006). Morley, Nelson, and Zivot (2003) (2003, henceforth MNZ) estimate a UC model with a stationary AR(2) cycle process, for U.S. quarterly real GDP. They show that one can identify and estimate the correlation parameter under this specification of cycle. Their estimate of the correlation parameter of –0.9062 is significantly different from zero according to their likelihood ratio test. They show also that the trend and cycle estimates with and without the zero correlation restriction are very different.

The main objective of this paper is to reinvestigate how shocks to these two components are correlated by applying unidentified UC models, in which a correlation is not identified unless we impose an identification restriction. We investigate how different identification restrictions lead to different values of the correlation between the two shocks. We also demonstrate how changing identification restriction results in different trend and cycle estimates. The empirical analysis in Section 4 shows that the trend and cycle estimates with different identification restrictions can be substantially different.

We call a UC model with ARMA(p, q) cycle a *UC-ARMA(p, q)* model. In explaining our methodology in Sections 2 and 3, we mainly consider a UC–ARMA(2, 1) model, for ease of exposition. The proposed methodology can, in principle, be extended to UC models with higher order ARMA cycles although the related calculations become more involved. In Section 1, first, we show that the UC–ARMA(2, 1) model is observationally equivalent to MNZ's UC–AR(2) model in the sense that these two UC models have the same autocovariance structure. A difficulty in applying the UC–ARMA(2, 1) model is that, unlike the UC–AR(2) model, it has a correlation parameter that *cannot* be identified (and hence estimated) unless we impose an identification restriction. Next, however, we show that for the correlation parameter, there is an upper bound implied by an unrestricted ARIMA(2, 1, 2) model, which is an observationally equivalent alternative representation of the UC–ARMA(2, 1) model. We propose a simple methodology for finding the implied upper bound. The basic idea of the methodology is to examine how the value of the correlation implied by an unrestricted ARIMA model changes when we impose different identification restrictions. In this way, we can obtain an implicit relationship between the identification restrictions and the resulting values of the correlation. [1] See Section 3 for more details.

The proposed methodology is applied to U.S. and U.K. real GDP data.[2] For both countries, it is found that the upper bounds of the correlations are negative. This implies that the two shocks are negatively correlated. We use UC-ARMA(2, 1) models, estimated under different identification restrictions, to estimate the trend and cycle. We find that estimates of the trend and cycle can vary substantially depending on the identification restrictions imposed. We find also that setting the MA(1) parameter equal to zero, or specifying the cycle as an AR(2) process, which has been one of the most commonly used specification for the cycle in the UC model literature, is not supported by the data on U.K. real GDP.

Our empirical analysis suggest that it is important to impose an appropriate identification restriction for properly estimating the trend and cycle. We discuss also on what is an appropriate identification restriction. In fact, it is confirmed empirically that the trend (and cycle) estimates obtained under (different but) appropriate identification restrictions are identical.

The rest of the paper is organized as follows. In the next section, we briefly overview the identification problem of UC models. In Section 3, we propose a simple methodology to find an upper bound of the correlation for unidentified UC models. In Section 4, we apply the proposed methodology to U.S. and U.K. real GDP data. The final section provides a summary and concluding remarks.

2 Overview of the identification problem of UC models

Let $\{y_t\}_{t=0}^{T}$ be an observed time series, such as the log of real GDP. We suppose that y_t is the sum of two unobserved stochastic processes, a random walk process τ_t and a stationary finite order ARMA $(p,\, q)$ process c_t; these processes are conventionally termed "trend" and "cycle", respectively, in the literature on business cycle analysis. The model is known as a UC model (hereafter, a UC-ARMA(p, q) model). Formally, the model is defined as follows:

$$\begin{aligned} &y_t = \tau_t + c_t, \quad \tau_t = \mu + \tau_{t-1} + \eta_t, \quad \phi(L)c_t = \theta(L)\varepsilon_t, \\ &\eta_t \sim i.i.d.(0, \sigma_\eta^2), \quad \varepsilon_t \sim i.i.d.(0, \sigma_\varepsilon^2), \quad \mathrm{cov}(\eta_t, \varepsilon_{t\pm s}) = \begin{cases} \sigma_{\eta\varepsilon} & \text{for} \quad s = 0, \\ 0 & \text{otherwise}, \end{cases} \end{aligned} \tag{1}$$

where $\phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p$ and $\theta(L) = 1 + \theta_1 L + \cdots + \theta_q L^q$ are pth order AR and qth order MA polynomials, respectively, that satisfy the stationarity and

[1] Oh, Zivot, and Creal (2008) do a similar analysis. They focus on comparing the estimates of trend and cycle obtained through the Beveridge-Nelson decomposition (Beveridge and Nelson, 1981) with the estimates of trend and cycle obtained from UC models with correlated shocks. By contrast, our focus is an investigation of the correlation.

[2] In the previous version of the paper, we do the same analysis to G7 countries except for Japan (Nagakura, 2007), and obtained results qualitatively similar to U.S. and U.K.

invertibility conditions; that is, the modulus of the roots of $\phi(z) = 0$ and $\theta(z)$ are all outside the unit circle.

From (1), we have

$$\phi(L)(1-L)y_t = \phi(1)\mu + \phi(L)\eta_t + (1-L)\theta(L)\varepsilon_t. \tag{2}$$

The right-hand side of (2) is the sum of two MA processes whose innovations are correlated.[3] However it can be shown that this part can be expressed by an MA (q^*) process with the single innovation u_t, where $q^* = \max\{p, q+1\}$ (see, for example, Granger and Morris, 1976). This implies that y_t can alternatively be represented as an ARIMA(p, q^*) process, as follows:

$$\phi(L)(1-L)y_t = \mu^* + \theta^*(L)\varepsilon_t, \quad u_t \sim i.i.d(0, \sigma_u^2), \tag{3}$$

where $\mu^* \equiv \phi(1)\mu$, and $\theta^*(L) \equiv 1 + \theta_1^* L + \cdots + \theta_{q^*}^* L^{q^*}$. Note that the AR coefficients in (2) and (3) are the same. This representation of the UC–ARMA(p, q) model is commonly referred to as the ARIMA (p, 1, q^*) reduced form.

MNZ point out that if one sets $p = 2$ and $q = 0$, then the parameters of the resulting UC–AR(2) model are uniquely identified from its ARIMA(2, 1, 2) reduced form parameters. To see this, let γ_j denote the jth order autocovariance of the MA part of (3). The first three autocovariances, γ_0, γ_1 and γ_2, in terms of the ARIMA(2, 1, 2) reduced form parameters, are given by $\gamma_0 = \sigma_u^2(1+\theta_1^{*2}+\theta_2^{*2})$, $\gamma_1 = \sigma_u^2\theta_1^*(1+\theta_2^*)$, $\gamma_2 = \sigma_u^2\theta_2^*$ and $\gamma_j = 0$ for $j \geq 3$. Given that the MA processes on the right-hand sides of (2) and (3) must be identical, we can express these autocovariances in terms of the UC model parameters, σ_η^2, σ_ε^2, and $\sigma_{\eta\varepsilon}$, as:

$$\begin{aligned}
\gamma_0 &= (1+\phi_1^2+\phi_2^2)\sigma_\eta^2 + 2\sigma_\varepsilon^2 + 2(1+\phi_1)\sigma_{\eta\varepsilon}, \\
\gamma_1 &= (\phi_1\phi_2 - \phi_1)\sigma_\eta^2 - \sigma_\varepsilon^2 + (\phi_2 - \phi_1 - 1)\sigma_{\eta\varepsilon}, \\
\gamma_2 &= -\phi_2\sigma_\eta^2 - \phi_2\sigma_{\eta\varepsilon}.
\end{aligned} \tag{4}$$

(See MNZ for a detailed derivation of the equations in (4)). Thus, given the ARIMA(2, 1, 2) reduced form parameters, which include ϕ_1 and ϕ_2, we can solve the three equations in (4) for the three unknown UC model parameters, σ_η^2, σ_ε^2, and $\sigma_{\eta\varepsilon}$, uniquely. The correlation ρ is calculated as $\rho = \sigma_{\eta\varepsilon}/(\sigma_\varepsilon\sigma_\eta)$.

A problem occurs when $p = 2$ and $q = 1$; then y_t follows a UC–ARMA(2, 1) process, and there is one additional parameter, namely, θ_1, the MA(1) coefficient of the ARMA(2, 1) cycle process. Although it is easy to show that its reduced form is also an ARIMA (2, 1, 2) process, the four UC model parameters, σ_η^2, $\sigma_{\eta\varepsilon}$, σ_η^2 and θ_1 cannot be uniquely identified from its ARIMA (2, 1, 2) reduced form parameters. To show this, we compare the autocovariances of the MA parts of the models. In terms of the UC model parameters, the autocovariances γ_j, $j = 0, 1, 2$ are:

$$\begin{aligned}
\gamma_0 &= (1+\phi_1^2+\phi_2^2)\sigma_\eta^2 + 2(1-\theta_1+\theta_1^2)\sigma_\varepsilon^2 + 2[1+\phi_1+\theta_1(\phi_2-\phi_1)]\sigma_{\eta\varepsilon}, \\
\gamma_1 &= (\phi_1\phi_2 - \phi_1)\sigma_\eta^2 - (1-\theta_1)^2\sigma_\varepsilon^2 + [\phi_2 - \phi_1 - 1 - \theta_1(\phi_2 - \phi_1 - 1)]\sigma_{\eta\varepsilon}, \\
\gamma_2 &= -\phi_2\sigma_\eta^2 - \theta_1\sigma_\varepsilon^2 - (\theta_1+\phi_2)\sigma_{\eta\varepsilon},
\end{aligned} \tag{5}$$

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[3] The sum of two MA processes with correlated innovations can be expressed as the sum of two MA processes with uncorrelated innovations. Let $\theta_i(L)\varepsilon_{i,t}$ $i = 1, 2$ be two MA processes with $\varepsilon_{i,t} \sim i.i.d.(0, \sigma_i^2)$ and $\mathrm{cov}(\varepsilon_{1,t}, \varepsilon_{2,t}) = \sigma$. Define $\widetilde{\varepsilon}_t \equiv \varepsilon_{2,t} - (\sigma/\sigma_1^2)\varepsilon_{1,t}$. Then, $\mathrm{cov}(\varepsilon_{1,t}, \widetilde{\varepsilon}_t) = 0$ and $\sum_{i=1}^2 \theta_i(L)\varepsilon_{i,t} = \widetilde{\theta}(L)\varepsilon_{1,t} + \theta_2(L)\widetilde{\varepsilon}_t$, where $\widetilde{\theta}(L) = \theta_1(L) + (\sigma/\sigma_1^2)\theta_2(L)$.

where θ_1 is the MA (1) coefficient of the ARMA(2, 1) cycle process.

Note that there are only three equations for the four unknown UC model parameters, σ_η^2, $\sigma_{\eta\varepsilon}$, σ_η^2 and θ_1. Thus, the equations in (5) cannot be uniquely solved for these four UC model parameters. To solve these equations, we must impose a restriction, which we term an "identification restriction", on θ_1. For example, setting $\theta_1 = 0$ reduces the model to a UC–AR(2) model. This restriction is not testable because under the alternative hypothesis the model parameters are not identified. In general, the order condition for identification of the UC–ARMA(p, q) model is satisfied when $p \geq q+2$. The model parameters are just identified when this equality holds. Thus, although UC-ARMA(p, $p-2$) and UC-ARMA(p, $p-1$) models both have an ARIMA(p, p) reduced form, the former model is identified while the latter model is not identified.

MNZ apply a UC–AR(2) model to U. S. quarterly real GDP data and find that the estimated correlation ρ is significantly negative. However, the point of our paper is that the true data generating process may be the UC–ARMA(2, 1) model that is observationally equivalent to the UC–AR(2) model. In that case, one cannot identify ρ.

3 Methodology

In this section, we illustrate a simple method for finding an upper bound for the correlation given the unrestricted ARIMA model parameters. The basic idea is to examine how the value of the correlation implied by an unrestricted ARIMA model changes when we impose different identification restrictions.

Lengthy calculations can be used to solve the equations in (5) to obtain the following expressions for the three UC model parameters:[4]

$$\sigma_\eta^2 = \frac{\gamma_0 + 2\gamma_1 + 2\gamma_2}{(1-\phi_1-\phi_2)^2},$$

$$\sigma_\varepsilon^2 = \frac{-2(1-\phi_1\theta_1+\phi_1+\phi_2\theta_1)(\gamma_2+\phi_2\sigma_\eta^2) - (\theta_1+\phi_2)[\gamma_0-(1+\phi_1^2+\phi_2^2)\sigma_\eta^2]}{2\theta_1(1-\phi_1\theta_1+\phi_1+\phi_2\theta_1) - 2(\theta_1+\phi_2)(1-\theta_1+\theta_1^2)}, \quad (6)$$

$$\sigma_{\eta\varepsilon} = \frac{\theta_1[\gamma_0-(1+\phi_1^2+\phi_2^2)\sigma_\eta^2] + 2(1-\theta_1+\theta_1^2)(\gamma_2+\phi_2\sigma_\eta^2)}{2\theta_1(1-\phi_1\theta_1+\phi_1+\phi_2\theta_1) - 2(\theta_1+\phi_2)(1-\theta_1+\theta_1^2)}.$$

Note that the variance of the trend shock, σ_η^2, is identified and is equivalent to the long-run variance of the first differences of $\{y_t\}$. This result holds in general: for

[4]Note that we can rewrite (5) as

$$\begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} 1+\phi_1^2+\phi_2^2 & 2(1-\theta_1+\theta_1^2 & 2[1+\phi_1+\theta_1(\phi_2-\phi_1) \\ \phi_1\phi_2-\phi_1 & -(1-\phi_1)^2 & [\phi_2-\phi_1-1-\theta_1(\phi_2-\phi_1-1)] \\ -\phi_2 & -\theta_1 & -(\theta_1+\phi_2) \end{bmatrix} \begin{bmatrix} \sigma_\eta^2 \\ \sigma_\varepsilon^2 \\ \sigma_{\eta\varepsilon} \end{bmatrix}.$$

We can show that the 3 by 3 matrix on the right hand side is non-singular if $\phi_1+\phi_2 \neq 1$, $\theta_1 \neq 1$, and $\theta_1(\theta_1+\phi_1) \neq \phi_2$ (and we assume it). Then we can uniquely solve the equation for σ_η^2, σ_ε^2, $\sigma_{\eta\varepsilon}$ as in (6).

any UC–ARMA(p, q) model, the variance of the trend shock is always identified as the long-run variance of the first differences; that is, $\sigma_\eta^2 = \psi(1)^2\sigma_u^2$, where $\psi(1) = \theta^*(1)/\phi(1)$. This was first pointed out by Cochrane (1988, p.908).

Note that given the ARIMA(2, 1, 2) reduced form parameters, the three UC model parameters above are functions of θ_1. Hence, given θ_1, these functions determine the values of the three UC model parameters that satisfy the equations in (5), from which we can calculate ρ. In this way, we can obtain an implicit relationship between the "identification restrictions" imposed on θ_1 and the resulting values of the correlation ρ. Figure 2(a) plots such pairs of values for θ_1 and ρ, given estimates of the ARIMA(2, 1, 2) model for U.S. quarterly real GDP, which are reported in the second column of Table 1.

The implied values of the correlations are all negative, and the upper bound of the correlation is around -0.75. The dashed line shows that if we restrict θ_1 to be 0, in which case the UC–ARMA(2, 1) model reduces to the UC–AR(2) model, then the resulting implied correlation is around -0.95, which is lower than the estimate obtained by MNZ. This is because our data are different from theirs; in particular, our data cover a longer sample period (more details on the data set are given in the next section). Note that there are ranges of values for θ_1 that imply correlations of less than -1, which thus violate the condition for positive definiteness of the covariance matrix. This means that UC–ARMA(2, 1) models with values of θ_1 in such ranges are inconsistent with the estimated unrestricted ARIMA(2, 1, 2) model. We refer to such values of θ_1 as "improper identification restrictions"; values of θ_1 at which $|\rho| \leq 1$ constitute "proper identification restrictions". In the figure, we display values of ρ based only on values of θ_1 in a particular range; this is because we confirmed that values for θ_1 outside of the range are inconsistent with $|\rho| \leq 1$.

In this way, we can find an upper bound for the correlation parameter. The methodology can be easily extended for other unidentified UC models.

4 Empirical applications

In this section, we apply the proposed methodology to U.S. and U.K. quarterly real GDP data. These quarterly real GDP data cover the period from 1946:4 to 2006:3 (yielding 238 observations) for the U.S. and the period from 1955:4 to 2006:2 (yielding 202 observations) for the U.K. For these periods, Figure 1 shows the percentage growth rates in GDP.

Table 1 reports the estimates of the unrestricted ARIMA (2, 1, 2) model for each (logged) GDP series. Figure 2 graphs the implied relationships between θ_1 and ρ for these GDP data, obtained by using the methodology described in the previous section. The implied values of the correlations are all negative for both countries. The upper bound for the correlation differs between the two countries; it is about -0.75 for U.S. and -0.993 for U.K. This implies that the two shocks are highly negatively correlated. Note that because these values are upper bounds, the actual correlations may be lower than these values. For the U.K. GDP data, the range of θ_1 values that satisfies the condition for positive definiteness of the the covariance matrix (i.e., $|\rho| \leq 1$) does not include $\theta_1 = 0$. This implies that, for the U.K., the restriction that $\theta_1 = 0$, under which the model reduces to the UC–AR(2) model, is inconsistent with the estimated unrestricted ARIMA models. In other words, the U.K. real GDP data do not support the UC – AR(2) specification.

The results show that for both countries, in particular U.K., the correlations of two shocks are highly negative. One hypothesis for explaining these strong negative correlations is that although we assumed that the real GDP is driven by two different shocks, namely, trend and cycle shocks, the real GDP is actually driven by only one shock that affects oppositely to trend and cycle. Another possible hypothesis is that the trend is not a random walk process. As shown in Nagakura (2008) and Nagakura and Zivot (2007), if the trend is not a random walk process but follows a certain class of I(1) process, then estimates of the correlation in the UC model in that the trend is assumed to be a random walk process tends to be negative. See Nagakura (2008) and Nagakura and Zivot (2007) for more details.

If we set the value of θ_1 *a priori* to, for example, $\theta_1 = 0$, we can directly estimate the other three UC model parameters from a state space representation.[5]. However, the above result suggests that we should not arbitrarily choose the value of θ_1. One would expect UC models estimated under different identification restrictions, particularly improper identification restrictions, to produce different trend and cycle estimates. To address this concern, we estimate the UC–ARMA(2, 1) model directly under different identification restrictions, including proper and improper restrictions, and then estimate the cycle and trend from these estimated UC models. In the estimation of UC models, we impose the positive definiteness condition on the covariance matrix parameters and impose the stationarity conditions on the AR(2) coefficients.

Table 2 reports the estimation results for the UC–ARMA(2, 1) model. The values of θ_1 in the first row are the restrictions imposed in advance. The asterisks denote improper restrictions. When we impose proper restrictions, the values of the log-likelihoods are the same as those of the unrestricted ARIMA model and are higher than those obtained under improper restrictions. Figures 3 and 4 display the cycle estimates for U.S. and U.K. real GDP, respectively. These are estimated by using Kalman filtering on the state space representation of the UC–ARMA(2, 1) model with estimated UC model parameters. For the U.S., Figures 3(a), (b) and (c) illustrate the cycle estimates from the UC models estimated under the restrictions $\theta_1 = 0$, $\theta_1 = -0.5$ and $\theta_1 = 0.5$, respectively. The restrictions, $\theta_1 = 0$ and $\theta_1 = -0.5$, are consistent with the estimated unrestricted ARIMA(2, 1, 2) model, whereas the restriction $\theta_1 = 0.5$ is not. The cycle estimates based on $\theta_1 = 0$ and $\theta_1 = -0.5$ are identical; however, the cycle estimates based on $\theta_1 = 0.5$ are substantially different from the other two. From Figure 4, findings for U.K. GDP are similar. Figures 4(a) and (b) illustrate the cycle estimates under proper identification restrictions and Figures 4(c) and (d) present the cycle estimates based on improper restrictions. The cycle estimates in (a) and (b) are identical. Although it is difficult to see visually, the cycle estimates in (c) differ from those in (a) and (b).

[5]From footnote 4, it is obvious that if θ_1 is fixed, γ_0, γ_1, and γ_2 are uniquely determined from σ_η^2, σ_ε^2, and $\sigma_{\eta\varepsilon}$, which implies that if θ_1 is fixed, we can estimate σ_η^2, σ_ε^2, and $\sigma_{\eta\varepsilon}$ by the MLE under Gaussian assumption for ε_t and η_t.

5 Conclusion

In this paper, we proposed a simple methodology for investigating the correlation between permanent and transitory shocks for unidentified UC models. Although one cannot estimate the correlation in this case, our methodology can be used to obtain an upper bound for the correlation. We applied our methodology to U.S. and U.K. real GDP data. It was found that the upper bounds of the correlations are negative for both countries. This implies that for these two countries, permanent and transitory shocks are strongly negatively correlated.

Our results raise questions about the conventional identification scheme for UC models, which involves setting the correlation parameter to zero. As argued by MNZ in the context of U.S. GDP, imposing such a restriction distorts the estimates of trend and cycle from UC models. Our results confirm this for the case of U.K. GDP. We also showed that the UC model with a stationary AR(2) cycle process is not supported by U.K. real GDP data.

Appendix A: State space representation of the UC–ARMA(2, 1) model

We adopt the following state space representation of the UC–ARMA(2, 1) model for estimation of the model parameters and the cycle:

(Observation equation)

$$y_t = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tau_t \\ c_t \\ c_{t-1} \\ \varepsilon_t \end{bmatrix},$$

(State equation)

$$\begin{bmatrix} \tau_t \\ c_t \\ c_{t-1} \\ \varepsilon_t \end{bmatrix} = \begin{bmatrix} \mu \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \phi_1 & \phi_2 & \theta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tau_{t-1} \\ c_{t-1} \\ c_{t-2} \\ \varepsilon_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_t \\ \varepsilon_t \end{bmatrix},$$

for $t = 1, ..., T$ with $E(\eta_t) = E(\varepsilon_t) = 0$, $\mathrm{var}(\eta_t) = \sigma_\eta^2$, $\mathrm{var}(\varepsilon_t) = \sigma_\varepsilon^2$, and $\mathrm{cov}(\eta_t, \varepsilon_s) = \rho\sigma_\eta\sigma_\varepsilon$ if $t = s$ and 0 otherwise. We set the initial conditions, namely, the mean vector and covariance matrix of $(\tau_1, c_1, c_0, \varepsilon_1)'$, for starting the Kalman filter recursion, as the stationary mean vector and covariance matrix of $(c_t, c_{t-1}, \varepsilon_t)'$ for $(c_1, c_0, \varepsilon_1)'$, and $E(\tau_1) = y_1$ and $\mathrm{var}(\tau_1) = 10^7$ for τ_1. The covariances between τ_1 and $(c_1, c_0, \varepsilon_1)'$ are set to zero.

If our objective is only to estimate parameters, it is more convenient to use the following state space representation of the first difference of y_t

(Observation equation)

$$\Delta y_t = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\tau_t \\ \Delta c_t \\ \Delta c_{t-1} \\ \varepsilon_t \\ \varepsilon_{t-1} \end{bmatrix},$$

(State equation)

$$\begin{bmatrix} \Delta\tau_t \\ \Delta c_t \\ \Delta c_{t-1} \\ \varepsilon_t \\ \varepsilon_{t-1} \end{bmatrix} = \begin{bmatrix} \mu \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \phi_1 & \phi_2 & \theta_1 - 1 & \theta_1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta\tau_{t-1} \\ \Delta c_{t-1} \\ \Delta c_{t-2} \\ \varepsilon_{t-1} \\ \varepsilon_{t-2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_t \\ \varepsilon_t \end{bmatrix}.$$

This is because then all state variables are stationary and we can avoid the problem of initialization. The same technique can be used for higher order UC-ARMA models. See Durbin and Koopman (2001) for more details of state space models.

References

Basistha, Arabinda (2007) Trend-cycle correlation, drift break and the estimation of trend and cycle in Canadian GDP. *Canadian Journal of Economics* Vol. 40, pp. 584-606.

Basisthat, Arabinda and Nelson, R. Charles (2007) New measures of the output gap based on the forward-looking new Keynesian Phillips curve. *Journal of Monetary Economics* Vol. 54(2), pp.498-511.

Beveridge, Stephen, and Chrles R. Nelson (1981) A New Approach to The Decomposition of Economic Time Series into Permanent And Transitory Components with Particular Attention to Measurement of The Business Cycle. *Journal of Monetary Economics* 7, 151–154.

Clark, Peter K. (1987) The Cyclical Component of U.S. Economic Activity. *The Quarterly Journal of Economics* 102, 797–814.

Cochrane, John H. (1988) How Big Is The Random Walk in GNP? *Journal of Political Economy* 96, 893–920.

Durbin, James, and Siem J. Koopman (2001) *Time Series Analysis by State Space Methods*. New York: Oxford University Press.

Fukuda, Kosei (2007) Are Trend and Cycle Innovations Uncorrelated? International Evidence *Applied Economics Letters(2007).* 14, 923–926.

Granger, Clive W. J. and M. J. Morris (1976) Time Series Modelling And Interpretation. *Journal of the Royal Statistical Society. Series A (General)* 139, 246–257.

Morley, James C. and Charles R. Nelson with Eric Zivot (2003) Why Are The Beveridge–Nelson And Unobserved Components Decompositions of GDP So Different? *The Review of Economics and Statistics* 85, 243–253.

Nagakura, D. (2007) Inference on The Correlation Between Permanent And Transitory Shocks in Unidentified Unobserved Components Models. Available at http://ssrn.com/abstract=981646

Nagakura, D. (2008) A Note on the Two Assumptions for Standard Unobserved Components Models. *Economics Letters*, Vol. 100(1), pp. 123-125.

Nagakura, D. and E. Zivot (2007) Implications of Two measures of Persistence for Correlation between Shocks to Permanent and Transitory Components in Unobserved Components Models, *University of Washington Working Papers* WEC-2007-07.

Oh, K., E., Zivot and D., Creal (2008) The Relationship Between The Beveridge–Nelson Decomposition And Unobserved Component Models with Correlated Shocks. *Journal of Econometrics*, Vol. 146(2), pp. 207-219.

Sinclair TM. 2009. The relationships between permanent and transitory movements in U.S. output and the unemployment rate. forthcoming, *Journal of Money, Credit, and Banking*.

Watson, Mark W. (1986) Univariate Detrending Methods with Stochastic Trends. *Journal of Monetary Economics* 18, 49–75.

Zarnowitz, Victor, and Ataman, Ozyildirim (2006) Time Series Decomposition And Measurement of Business Cycles, Trends And Growth Cycles. *Journal of Monetary Economics* 53, 1717–1739.

Table 1: Estimates of the ARIMA (2, 1, 2) parameters

	U.S.	U.K.
ϕ_1	1.3635	0.5605
	(0.1447)	(0.0972)
ϕ_2	–0.7789	–0.2564
	(0.1729)	(0.0981)
θ_1^*	–1.1068	–0.1361
	(0.2148)	(0.0651)
θ_2^*	0.6187	0.7560
	(0.2234)	(0.0692)
μ	0.8299	0.6189
	(0.0725)	(0.0661)
σ_u^2	0.8253	0.1645
σ_{lrv}^2	1.253	0.892
σ_{ucv}^2	0.970	0.301
$\psi(1)$	1.2293	2.3278
L	–315.04	–105.14

Note: the following ARIMA(2, 1, 2) model was estimated by using exact maximum likelihood estimation:

$$\phi(L)(\Delta y_t - \mu) = \theta(L)u_t, \quad u_t \sim NID(0, \sigma_u^2),$$

where $\phi(L) = 1 - \phi_1 L - \phi_2 L^2$, and $\theta^*(L) = 1 + \theta_1^* L + \theta_2^* L^2$. Standard errors are in parentheses. The rows with σ_{lrv}^2, σ_{ucv}^2 and $\psi(1)$ display estimates of the long-run variance, the unconditional variance and the (cumulated) impulse response measure, namely, $\psi(1) = \theta^*(1)/\phi(1)$, respectively. The last row reports the log-likelihood.

Table 2: Estimates of the UC–ARMA(2, 1) parameters

	U.S.			U.K.			
θ_1	0.0	–0.5	0.5*	0.16	0.22	0.0*	−0.5*
ϕ_1	1.3635	1.3635	1.4277	0.5605	0.5605	0.6462	1.6709
ϕ_2	−0.7789	−0.7789	−0.6051	−0.2564	−0.2564	−0.3342	−0.7547
σ_η^2	1.2533	1.2533	0.8733	0.8914	0.8914	0.8751	0.7535
σ_ϵ^2	0.3170	0.3798	0.4211	0.3276	0.3780	0.2864	0.7405
ρ	−0.9483	−0.7429	−0.9517	−0.9937	−0.9948	−0.9983	−0.9381
μ	0.8299	0.8299	0.8337	0.6189	0.6189	0.6186	0.6192
$\sigma_\eta/\sigma_\epsilon$	1.9883	1.8165	2.0741	1.6495	1.5356	3.0557	1.0087
L	−315.04	−315.04	−316.69	−105.14	−105.14	−105.63	−111.35

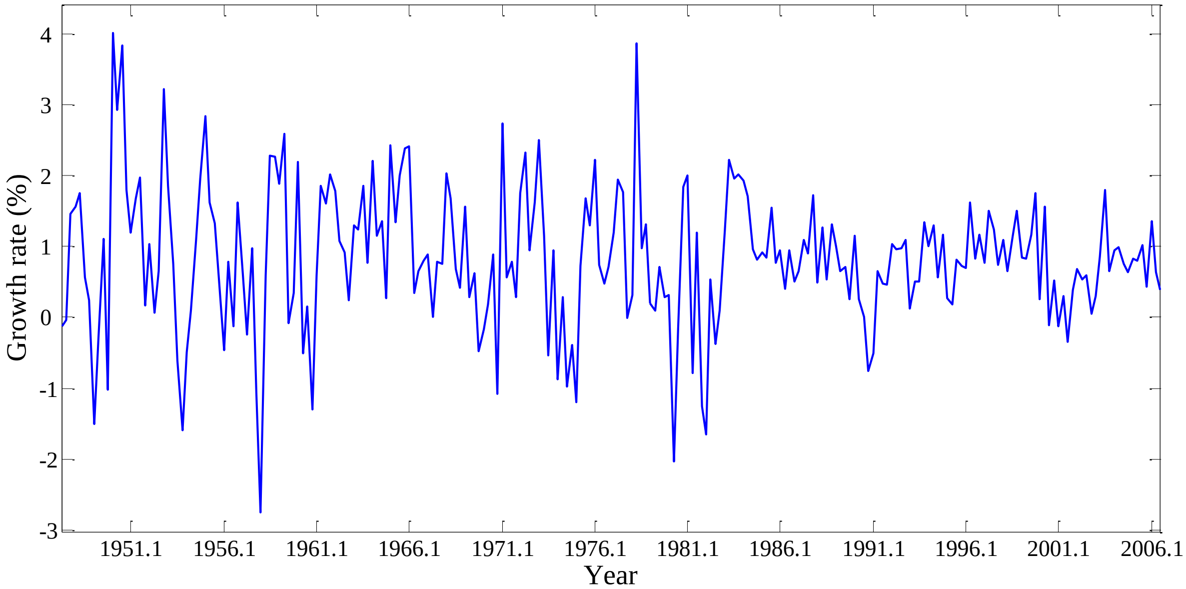
Note: the following UC–ARMA(2, 1) model was estimated by maximum likelihood estimation from a state space representation :

$$y_t = \tau_t + c_t, \quad \tau_t = \mu + \tau_{t-1} + \eta_t, \quad \phi(L)c_t = \theta(L)\varepsilon_t$$
$$\eta_t \sim i.i.d.N(0, \sigma_\eta^2), \quad \epsilon_t \sim i.i.d.N(0, \sigma_\varepsilon^2), \quad corr(\eta_t, \varepsilon_{t\pm k}) = \begin{cases} \rho\sigma_\eta\sigma_\varepsilon & \text{for} \quad k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

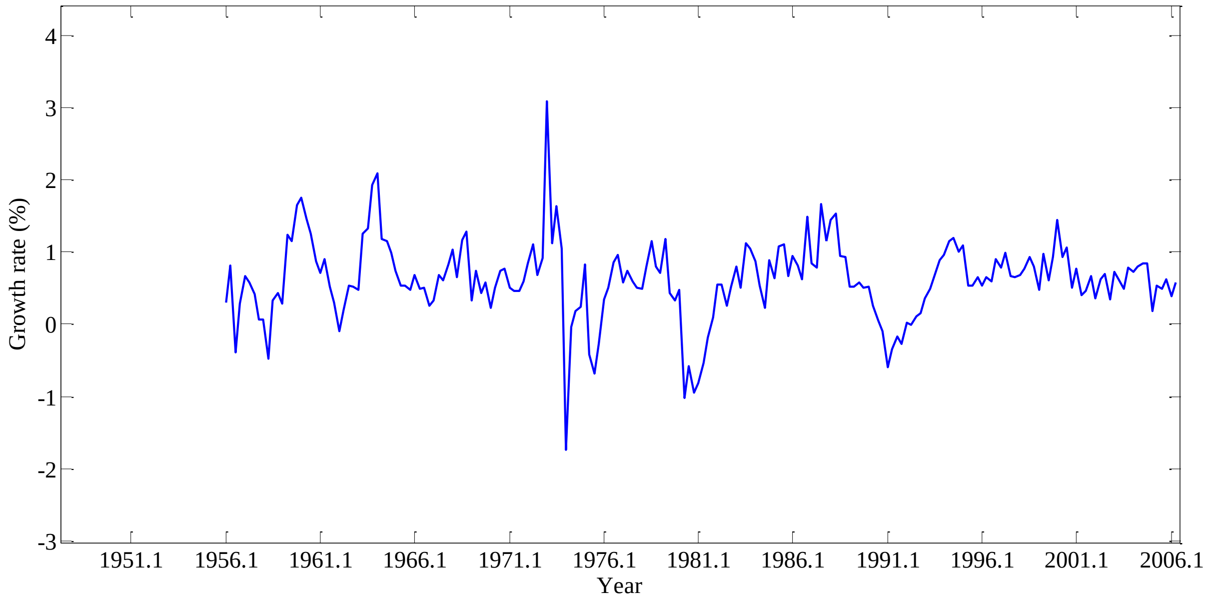
where $\phi(L) = 1 - \phi_1 L - \phi_2 L^2$, and $\theta(L) = 1 + \theta_1 L$. The value of θ_1 was set before estimation. For estimation, we imposed the condition for positive definiteness on the covariance matrix parameters and imposed the stationarity conditions for the AR(2) coefficients. The last row reports the value of the log-likelihood.

Figure 1: Growth rates of U.S. and U.K. real GDP

(a) U.S.



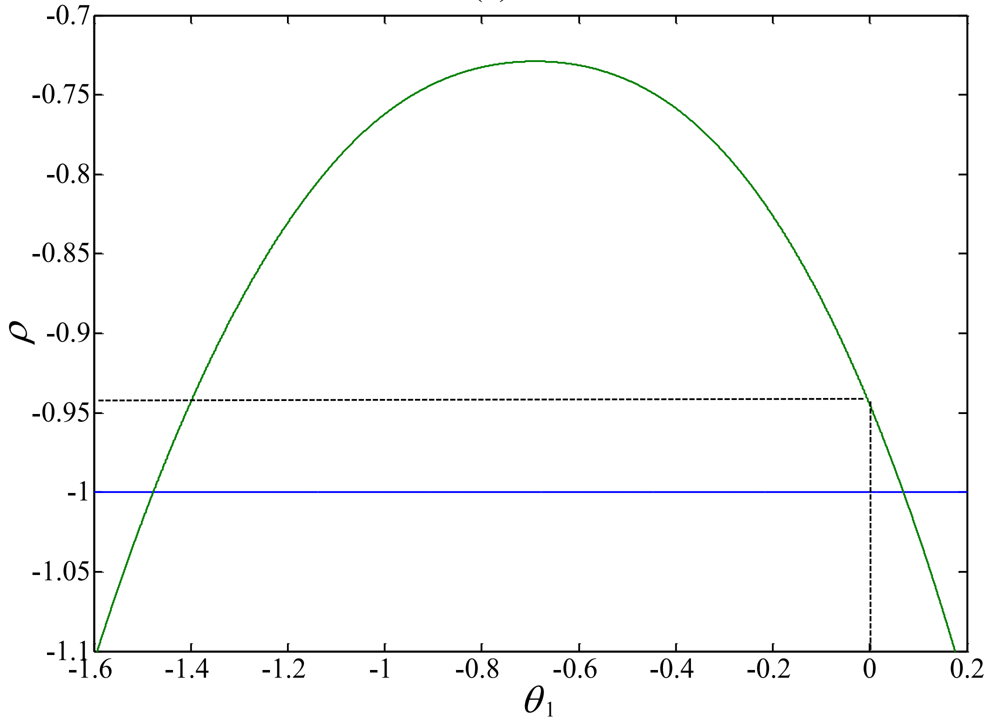
(b) U.K.



U.S.:1947.1-2006.3 (238 observations)   U.K.:1956.1-2006.2 (202 observations)

Figure 2: Implied relationship between the correlation and the MA(1) parameter

(a) U.S.



(b) U.K.

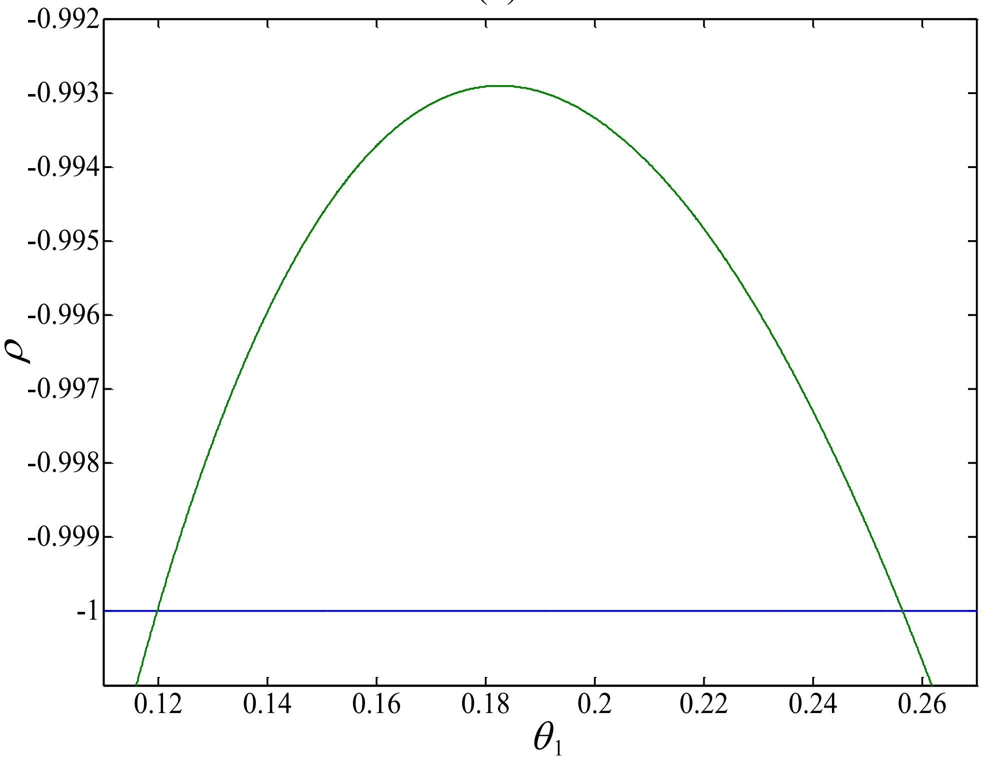
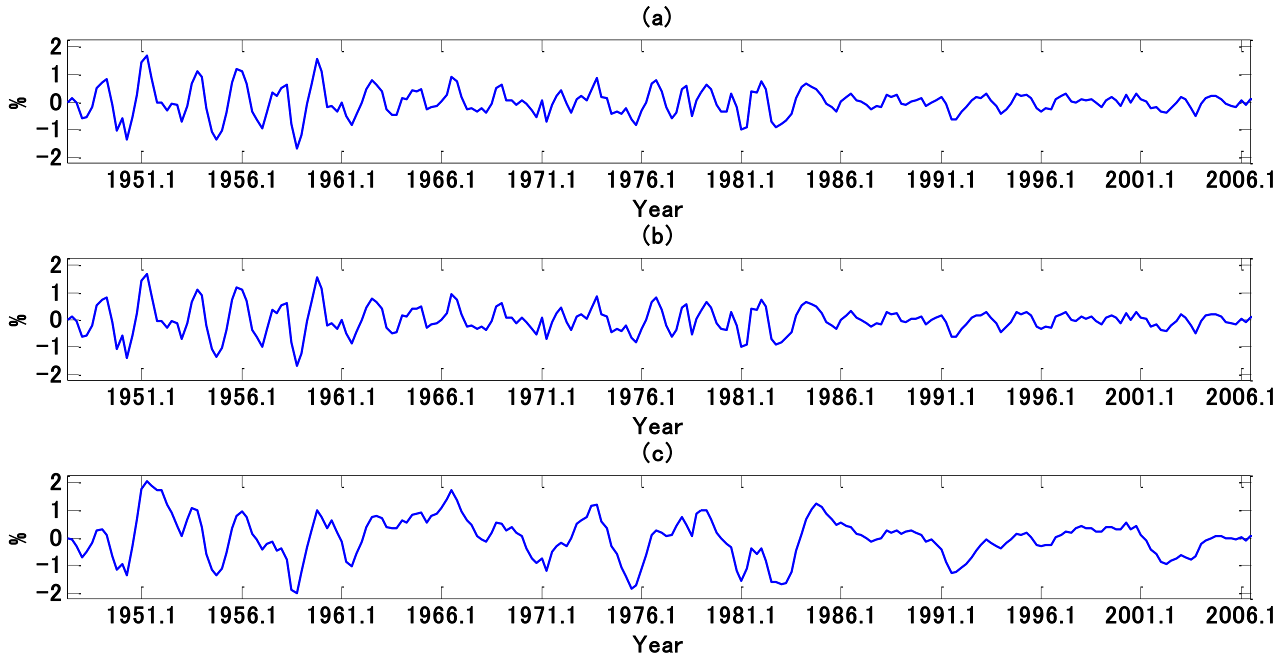
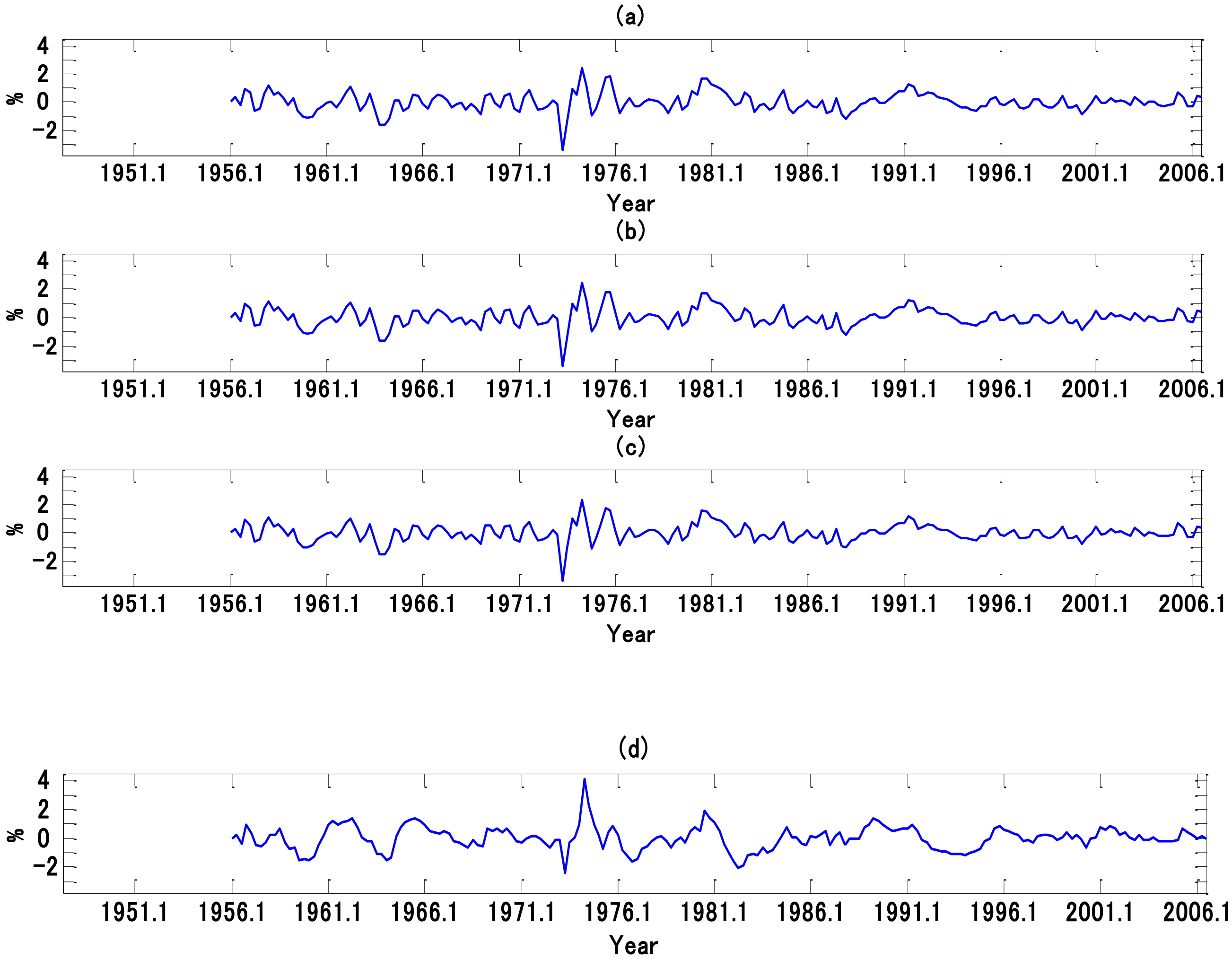


Figure 3: Percentage deviation from trend of U.S. real GDP



Note: these figures represent the cycle estimates from the UC models estimated under the following restrictions on $\theta_1 : \theta_1 = 0$ for (a); $\theta_1 = 0.5$ for (b); and $\theta_1 = -0.5$ for (c).

Figure 4: Percentage deviation from trend of U.K. real GDP



Note: these figures represent the cycle estimates from the UC models estimated under the following restrictions on $\theta_1 : \theta_1 = 0.16$ for (a); $\theta_1 = 0.22$ for (b); $\theta_1 = 0$ for (c) and $\theta_1 = -0.5$ for (d)