Hitotsubashi University

Global COE Hi-Stat Discussion Paper Series 281

Research Unit for Statistical and Empirical Analysis in Social Sciences (Hi-Stat)

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February 2013

Hi-Stat Discussion Paper

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Modelling for the wavelet coefficients of ARFIMA processes

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January 28, 2013

Abstract

We consider the model for the discrete nonboundary wavelet coefficients of ARFIMA processes. Although many authors have explained the utility of the wavelet transform for the long dependent processes in semiparametrical literature, there have been a few studies in parametric setting. In this paper, we restrict the Daubechies wavelets filters to make the form of the (general) spectral density function of these coefficients clear.

Key Words : discrete wavelet transform; long memory process; spectral density function

JEL classification : C22

1 Introduction

What kind of models is the process of the wavelet coefficients of autoregression fractionally integrated moving-average (ARFIMA) processes? This question is simple, but complex, which we face when we use the wavelet method to ARFIMA processes. Many authors have explained in semiparametrical literature the reason why the covariances among the wavelet coefficients of the long memory processes can be assumed zero to construct the wavelet-based estimation procedures for the memory parameter, d. On the other hands, in order to improve

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the estimator, there have been some studies in which the dependency between the wavelet transformed long memory process are taken in consideration. For example, Jensen (2000) computed the covariance matrix of the wavelet coefficients of the stationary ARFIMA model, and Craigmile, Guttorp, and Percival (2005) applied an AR(1) model to the nonboundary wavelet coefficients in each scale. The ARFIMA model is one of the K-th order differenced processes and introduced by Granger and Joyeux (1980) and Hosking (1981), which is a generalization of the standard ARIMA(p,d,q,) models.

Let $\{X_t\}$ be ARFIMA(p,d,q) process, then

$$\Phi_p(B)(1-B)^d X_t = \Theta_q(B)\varepsilon_t, \tag{1}$$

where B is a back shift operator, i.e., $BX_t = X_{t-1}$,

$$\Phi_p(B) = 1 - \sum_{k=1}^{p} \phi_k B^k, \tag{2}$$

and

$$\Theta_q(B) = 1 + \sum_{k}^{q} \theta_k B^k, \tag{3}$$

with $\Theta_q(0) \neq 0$. $\Phi_p(B)$ and $\Theta_q(B)$ are real polynomials of degrees p and q, which share no common zeros, and all of their zeros are outside the unit circle in the complex plane. Then we can define its generalized spectral density function (SDF);

$$S_X(f) = \frac{\sigma_\varepsilon^2}{|2\sin(\pi f)|^{2d}} \frac{|\Theta_q(e^{i2\pi f})|^2}{|\Phi_p(e^{i2\pi f})|^2}, \quad |f| \leq \frac{1}{2}. \tag{4}$$

In this paper, we explore the model for the wavelet transformed ARFIMA processes by using their generalized SDF defined by eq.(4). This is the similar way to Krim and Pesquet (1995) where the discrete wavelet transformed coefficients of the process with $d \in \mathbb{N}$ are considered, the wavelet filter is not specified and the degree of the MA term is counted indirectly. In contrast to Krim and Pesquet (1995), we assume that the wavelet filter is one of Daubechies filters which is denoted as $D(L)$ and $LA(L)$ in Percival and Walden (2000) which have the same squared gain functions, in order to represent the SDF of the j-th nonboundary wavelet coefficients directly and explore its model when $d \in \mathbb{R}$ and the length of filters is L.

This paper is organized as follows; the notations and definitions of the discrete wavelet transform (DWT) are in section 2; the main results, theorems and remarks, are in section 3; the conclusion is in section 4, and the proofs of theorems and lemmas are in the appendix.

2 Notations and definitions

In this section, we prepare the notations and definitions of the DWT which are the similar as in Percival and Walden (2000). The j-th level DWT consists of two parts; filtering the original process by j-th level wavelet and scaling filters, and 2^j downsampling these filtered processes.

First, let $\{h_l\}_{l=0}^{L-1}$ denote the discrete wavelet filter coefficients of a Daubechies compactly supported wavelet with the length L and let $\{g_l\}_{l=0}^{L-1}$ be the corresponding scaling filter coefficients which are defined by

$$g_l \equiv (-1)^{l+1} h_{L-1-l}.$$

Let's set $h_{1,l} = h_l$, $g_{1,l} = g_l$ and $L_1 = L$, then we can obtain the j-th level wavelet and scaling filter coefficients, $\{h_{j,l}\}_{l=0}^{L_j-1}$ and $\{g_{j,l}\}_{l=0}^{L_j-1}$, for $j \geq 2$, by

$$h_{j,l} = \sum_{k=0}^{L-1} h_{1,k} g_{j-1,l-2^{j-1}k} \quad \text{and} \quad g_{j,l} = \sum_{k=0}^{L-1} g_{1,k} g_{j-1,l-2^{j-1}k}, \tag{5}$$

where $L_j = (2^j - 1)(L-1) + 1$. These filters have the following properties;

$$\sum_{l=0}^{L_j-1} h_{j,l} = 0, \qquad \sum_{l=0}^{L_j-1} h_{j,l}^2 = 1,$$
$$\sum_{l=0}^{L_j-1} g_{j,l} = 2^{j/2}, \qquad \sum_{l=0}^{L_j-1} g_{j,l}^2 = 1,$$

and

$$\sum_{l=0}^{L_j-1} g_{j,l} h_{j,l} = 0.$$

The (j,k)-th wavelet and scaling coefficients of the series $\{a_t\}_{t=0}^{T-1}$ are computed as

$$W_{a,j,k} = \sum_{l=0}^{L_j-1} h_{j,l} a_{2^j(k+1)-l-1 \bmod N}, \tag{6}$$

and

$$V_{a,j,k} = \sum_{l=0}^{L_j-1} g_{j,l} a_{2^j(k+1)-l-1 \bmod N}. \tag{7}$$

where $L_1 = L$ and $L_j = (2^j - 1)(L-1)+1$. Using the relations of eq.(5) and setting $V_{a,0,t} = a_t$, we can calculate the (j,k)-th wavelet and scaling coefficients iteratively as follows;

$$W_{a,j,k} = \sum_{l=0}^{L-1} h_l V_{a,j-1,2k+1-l \mod N_{j-1}}, \tag{8}$$

and

$$V_{a,j,k} = \sum_{l=0}^{L-1} g_l V_{a,j-1,2k+1-l \mod N_{j-1}}. \tag{9}$$

These equations are so called "the pyramid algorithm". The pyramid algorithm which is introduced by Mallat (1989) is the elegant algorithm to compute the DWT and its inverse quickly. In practice, the larger the level is, the smaller the number of the wavelet coefficients in the level are.

To prove the theorems in the next section, we represent the spectral density function of the wavelet transformed process and need the squared gain functions of wavelet and scaling filters. First, the transfer functions of $\{h_l\}$ and $\{g_l\}$ are defined as

$$H_{1,L}(f) = \sum_{l=0}^{L-1} h_l e^{i2\pi f}, \tag{10}$$

and

$$G_{1,L}(f) = \sum_{l=0}^{L-1} g_l e^{i2\pi f}. \tag{11}$$

When we use the Daubechies wavelet filter, these squared gain functions, i.e., $\mathcal{H}_{1,L}(f) = |H_{1,L}(f)|^2$ and $\mathcal{G}_{1,L}(f) = |G_{1,L}(f)|^2$, are

$$\mathcal{H}_{1,L}(f) = 2\sin^L(\pi f) \sum_{l=0}^{\frac{L}{2}-1} \begin{pmatrix} \frac{L}{2} - 1 + l \\ l \end{pmatrix} \cos^{2l}(\pi f), \tag{12}$$

and

$$\mathcal{G}_{1,L}(f) = 2\cos^L(\pi f)\sum_{l=0}^{\frac{L}{2}-1}\left(\begin{array}{c}\frac{L}{2}-1+l\\ \\ l\end{array}\right)\sin^{2l}(\pi f), \tag{13}$$

For $j \geq 2$, the transfer functions of $\{h_{j,l}\}$ and $\{g_{j,l}\}$ are

$$H_{j,L}(f) = \sum_{l=0}^{L_j-1} h_{j,l}e^{i2\pi f}, \tag{14}$$

and

$$G_{j,L}(f) = \sum_{l=0}^{L_j-1} g_{j,l}e^{i2\pi f}. \tag{15}$$

We can also calculate them as

$$\mathcal{H}_{j,L}(f) = \mathcal{H}_{1,L}(2^{j-1}f)\prod_{l=0}^{j-2}\mathcal{G}_{1,L}(2^l f), \tag{16}$$

and

$$\mathcal{G}_{j,L}(f) = \prod_{l=0}^{j-1}\mathcal{G}_{1,L}(2^l f). \tag{17}$$

To simplify the argument, in the rest of paper, we only consider the nonboundary coefficients which are not affected by the boundary condition, i.e., the periodic calculation, of DWT, because it is difficult to evaluate the SDF of the boundary coefficients and the wavelet-based estimators for the memory parameter usually consist of only nonboundary coefficients.

3 Main results

Using the assumption of ARFIMA process and the notations and definitions of DWT in the previous section, we have the following theorem.

Theorem 1

Under the assumptions that $\{X_t\}$ is ARFIMA(p,d,q) and the $D(L)$ or $LA(L)$ wavelet filter is used, the process of the j-th level nonboundary wavelet coefficients of $\{X_t\}$ becomes the following model;

(i) if $L/2 - d \in \mathbb{Z}_+ \cup \{0\}$*, ARMA(*$p$*,*$q_j$*) with*

$$q_j \le p + L - 1 + (q - p - d - L + 1)/2^j;$$

(ii) if $L/2 - d \in \mathbb{R}_+ \setminus \mathbb{Z}_+$*, ARMA(*$p$*,*$\infty$*);*

(iii) if $L/2 - d \in \mathbb{Z}_-$*, ARFIMA(*p*,*$d - L/2$*,*q_j*) with*

$$q_j \le d + p + \frac{L}{2} - 1 + (q - p - d - L + 1)/2^j;$$

(iv) if $L/2 - d \in \mathbb{R}_- \setminus \mathbb{Z}_-$*, ARFIMA(*$p$*,*$d - L/2$*,*$\infty$*).*

The proof is in the appendix.

Remark 1: In case (i) and (ii) of 1, i.e., $L/2 > d$, the wavelet transformed process becomes not the negative memory process but the stationary ARMA process. On the other hand, in case (iii) and (iv) of 1, i.e., $L/2 < d$, the process becomes weaker long memory process than before.

Remark 2: The ARMA term in case (ii) and (iv) of 1 can be represent as AR(∞). Under the setting that $d < L/2$ and $p = q = 0$, i.e., the process is $I(d)$, Craigmile, Guttorp, and Percival (2005) compares the AR(p) approximations by numerical simulation and recommends the AR(1) approximation.

On a parallel with the above theorem, we can consider the model of the j-th level nonboundary scaling coefficients of ARFIMA(p,d,q).

Corollary 1

Under the same assumptions as in Theorem 1, the process of the j*-th level nonboundary scaling coefficients of* $\{X_t\}$ *becomes the following model;*

1. *if* $d \in \mathbb{Z}$*, ARFIMA(*p*,*d*,*$\tilde{q}_j$*) with*

$$\tilde{q}_j \le p + d + L - 1 + (q - p - d - L + 1)/2^j;$$

2. *if* $d \in \mathbb{R} \setminus \mathbb{Z}$, *ARFIMA(p,d,∞).*

This corollary simply explains that the scaling coefficients are sill long memory process.

We have a spacial case when the process $\{X_t\}$ is white noise, i.e., $p = q = d = 0$;

Corollary 2

When $\{X_t\}$ *is white noise, i.e., its SDF is defined as* $S_X(f) = \sigma_X^2$ *for* $|f| \leq \frac{1}{2}$*, then the SDFs of its* j*-th nonboundary wavelet and scaling coefficients are*

$$S_{W,j}(f) = \sigma_X^2, \quad \text{and} \quad S_{V,j}(f) = \sigma_X^2. \tag{18}$$

So these coefficients become white noises.

4 Conclusion

We have explained the effect of DWT for long memory process in parametric literature and shown the model of the wavelet transformed ARFIMA process. To use those results in practice, i.e, to estimate the parameters via the wavelet-based maximal likelihood, we have further works; we need to check that the MA(∞) or ARMA(p, ∞) term may be approximated to the AR($\tilde{p}$) with some integer $\tilde{p}$; we need to compare these performances with the performance under the white noise approximation.

ACKNOWLEDGMENT

The author thanks to Katsuto Tanaka, Toshiaki Watanabe and the members of their seminars. Financial support from the Ministry of Education, Culture, Sports, Science and Technology of the Japanese Government through the Global COE program "Research Unit for Statistical and Empirical Analysis in Social Sciences" at Hitotsubashi University is gratefully acknowledged.

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A The proofs and the technical lemma

A.1 Lemma

Lemma 1

For $j = 1, 2, \dots,$ *and* $\alpha \in \mathbb{R}$,

$$\prod_{k=0}^{2^j-1} |1 - e^{i2\pi k/2^j} e^{i2\pi f/2^j}|^{2\alpha} = |1 - e^{i2\pi f}|^{2\alpha}. \tag{19}$$

We show this lemma by the mathematical induction. **Proof.**

(i) For $j = 1$,

$$\prod_{k=0}^{1} |1 - e^{i2\pi k/2} e^{i2\pi f/2}|^{2\alpha} = |1 - e^{i2\pi f}|^{2\alpha}.$$

So eq.(19) is true for $j = 1$.

(ii) For $j = 2, 3, \dots,$ we assume that eq.(19) is true for $j - 1$, so it is true that

$$\prod_{k=0}^{2^{j-1}-1} |1 - e^{i2\pi k/2^{j-1}} e^{i2\pi f/2^{j-1}}|^{2\alpha} = |1 - e^{i2\pi f}|^{2\alpha}. \tag{20}$$

Now for $j = 2, 3, \dots,$

$$\begin{aligned}\prod_{k=0}^{2^j-1} |1 - e^{i2\pi k/2^j} e^{i2\pi f/2^j}|^{2\alpha} &= \prod_{k=0}^{2^{j-1}-1} |1 - e^{i2\pi k/2^j} e^{i2\pi f/2^j}|^{2\alpha} \\ &\times \prod_{k=0}^{2^{j-1}-1} |1 - e^{i2\pi (k+2^{j-1})/2^j} e^{i2\pi f/2^j}|^{2\alpha}.\end{aligned}$$

Because $(k + 2^{j-1})/2^j = k/2^j + 1/2$,

$$\prod_{k=0}^{2^j-1} |1 - e^{i2\pi k/2^j} e^{i2\pi f/2^j}|^{2\alpha} = \prod_{k=0}^{2^{j-1}-1} |1 - e^{i2\pi k/2^{j-1}} e^{i2\pi f/2^{j-1}}|^{2\alpha}.$$

Under the assumption of eq.(20), eq.(19) is true for $j = 2, 3, \dots.$

From the arguments of (i) and (ii), the lemma is proven. □

A.2 Proof of Theorem 1

A.2.1 Proof of (i) and (ii) of Theorem 1

For $j = 1$, the SDF of the 1-st level non-decimated nonboundary wavelet coefficients becomes

$$S_{\breve{W},1}(f) = 2^{1-L}\sigma_\varepsilon^2|2\sin(\pi f)|^{L-2d}\sum_{l=0}^{\frac{L}{2}-1} b_l|1+e^{i2\pi f}|^{2l}\frac{|\Theta_q(e^{i2\pi f})|^2}{|\Psi_p(e^{i2\pi f})|^2},$$

where

$$a_l \stackrel{\text{def}}{=} \begin{pmatrix} \frac{L}{2}-1+l \\ l \end{pmatrix} \quad \text{and} \quad b_l \stackrel{\text{def}}{=} a_l 2^{-2l},$$

for $l = 0, 1, \ldots, \frac{L}{2}-1$. Obviously, $b_0 = 1$.

The SDF of the 1-st level nonboundary wavelet coefficients is computed by

$$S_{W,1}(f) = \frac{1}{2}\left[S_{\breve{W},1}\left(\frac{f}{2}\right) + S_{\breve{W},1}\left(\frac{f+1}{2}\right)\right],$$

and we set

$$S_{W,1}(f) \stackrel{\text{set}}{=} \frac{|\widehat{\Theta}_1(e^{i2\pi f})|^2}{|\widehat{\Psi}_1(e^{i2\pi f})|^2},$$

where

$$\begin{aligned} |\widehat{\Psi}_1(e^{i2\pi f})|^2 &= |\Psi_p(e^{i2\pi f/2})\Psi_p(-e^{i2\pi f/2})|^2, \\ |\widehat{\Theta}_1(e^{i2\pi f})|^2 &= \frac{1}{2}\left[|\widehat{\Theta}_{1,0}(e^{i2\pi f})|^2 + |\widehat{\Theta}_{1,1}(e^{i2\pi f})|^2\right], \end{aligned}$$

with

$$\begin{aligned} &|\widehat{\Theta}_{1,0}(e^{i2\pi f})|^2 \\ &\quad = 2^{1-L}\sigma_\varepsilon^2|1-e^{i2\pi f/2}|^{L-2d}\left[1+\sum_{l=1}^{\frac{L}{2}-1} b_l|1+e^{i2\pi f/2}|^{2l}\right]|\Theta_q(e^{i2\pi f/2})|^2|\Psi_p(-e^{i2\pi f/2})|^2, \end{aligned}$$

and

$$\begin{aligned} &|\widehat{\Theta}_{1,1}(e^{i2\pi f})|^2 \\ &\quad = 2^{1-L}\sigma_\varepsilon^2|1+e^{i2\pi f/2}|^{L-2d}\left[1+\sum_{l=1}^{\frac{L}{2}-1} b_l|1-e^{i2\pi f/2}|^{2l}\right]|\Theta_q(-e^{i2\pi f/2})|^2|\Psi_p(e^{i2\pi f/2})|^2. \end{aligned}$$

If $\widehat{\Theta}_1(Z)=0$ has roots on the unit-circle, these roots make simultaneously $\widehat{\Theta}_{1,k}(Z)=0$ for $k=0,1$. But there is no root satisfying this condition, and then $\widehat{\Theta}_1(Z)$ is a stationary MA term. When $L/2-d\in\mathbb{Z}_+$, the upper bound of the degree of this MA term is

$$\left(\frac{L}{2}-d\right)/2+\left(\frac{L}{2}-1\right)/2+q/2+p/2=(L-1)/2+(q+p-d)/2.$$

For $j\geq 2$, the generalized SDF of the j-th level non-decimated nonboundary wavelet coefficients is

$$\begin{aligned}
&\mathcal{H}_{j,L}(f)S_X(f)\\
&=2^{1-L}\sigma_\varepsilon^2|1-e^{i2\pi f}|^{L-2d}\left[\sum_{l=0}^{\frac{L}{2}-1}b_l|1+e^{i2\pi f2^{j-1}}|^{2l}\right]\\
&\times\prod_{m=0}^{j-2}\left[2^{1-L}|1+e^{i2\pi f2^m}|^{2L}\left\{\sum_{l_m=0}^{\frac{L}{2}-1}b_{l_m}|1-e^{i2\pi f2^m}|^{2l_m}\right\}\right]\\
&\times\frac{|\Theta_q(e^{i2\pi f})|^2}{|\Psi_p(e^{i2\pi f})|^2}.
\end{aligned}$$

Since the j-th level nonboundary wavelet coefficients are 2^j-downsampled series of the j-th level non-decimated nonboundary wavelet coefficients, its generalized SDF becomes

$$S_{W,j}(f)=2^{-j}\sum_{k=0}^{2^j-1}\mathcal{H}_{j.L}\left(\frac{f+k}{2^j}\right)S_X\left(\frac{f+k}{2^j}\right),$$

where

$$\begin{aligned}
&\mathcal{H}_{j,L}\left(\frac{f+k}{2^j}\right)S_X\left(\frac{f+k}{2^j}\right)\\
&=2^{1-L}\sigma_\varepsilon^2|1-e^{i2\pi k2^{-j}}e^{i2\pi f2^{-j}}|^{L-2d}\left[\sum_{l=0}^{\frac{L}{2}-1}b_l|1+e^{i2\pi\frac{k}{2}}e^{i2\pi\frac{f}{2}}|^{2l}\right]\\
&\times\prod_{m=0}^{j-2}\left[2^{1-L}|1+e^{i2\pi\frac{k}{2^{j-m}}}e^{i2\pi\frac{f}{2^{j-m}}}|^{2L}\left\{\sum_{l_m=0}^{\frac{L}{2}-1}b_{l_m}|1-e^{i2\pi\frac{k}{2^{j-m}}}e^{i2\pi\frac{f}{2^{j-m}}}|^{2l_m}\right\}\right]\\
&\times\frac{|\Theta_q(e^{i2\pi\frac{k}{2^j}}e^{i2\pi\frac{f}{2^j}})|^2}{|\Psi_p(e^{i2\pi\frac{k}{2^j}}e^{i2\pi\frac{f}{2^j}})|^2}.
\end{aligned}$$

This MA term can be set as follows;

$$\left|\widehat{\Theta}_j(Z)\right|^2\overset{\text{set}}{=}2^{-j}\sum_{k=0}^{2^j-1}\left|\widehat{\Theta}_{j,k}(Z)\right|^2,$$

where

$$\left|\widehat{\Theta}_{j,k}(Z)\right|^2 \stackrel{\text{def}}{=} 2^{1-L}\sigma_\varepsilon^2|1-e^{i2\pi k2^{-j}}Z^{2^{-j}}|^{L-2d}\left[\sum_{l=0}^{\frac{L}{2}-1} b_l|1+e^{i2\pi\frac{k}{2}}Z^{\frac{1}{2}}|^{2l}\right]$$
$$\times\prod_{m=0}^{j-2}\Bigg[2^{1-L}|1+e^{i2\pi\frac{k}{2^{j-m}}}Z^{\frac{1}{2^{j-m}}}|^{2L}$$
$$\times\left\{\sum_{l_m=0}^{\frac{L}{2}-1} b_{l_m}|1-e^{i2\pi\frac{k}{2^{j-m}}}Z^{\frac{1}{2^{j-m}}}|^{2l_m}\right\}\Bigg]$$
$$\times|\Theta_q(e^{i2\pi\frac{k}{2^j}}Z^{\frac{1}{2^j}})|^2\prod_{h=0,h\neq k}^{2^j-1}|\Psi_p(e^{i2\pi\frac{h}{2^j}}Z^{\frac{1}{2^j}})|^2.$$

There is no root on unit-circle which satisfies $\widehat{\Theta}_j(Z)=0$, and, when $\frac{L}{2}-d\in\mathbb{Z}_+$, the upper bound of the degree of the MA term is

$$2^{-j}\left(\frac{L}{2}-d\right)+\frac{1}{2}\left(\frac{L}{2}-1\right)+\left(\frac{3}{2}L-1\right)2^{-j}\sum_{m=0}^{j-2}2^m+q2^{-j}+(2^j-1)p2^{-j}$$
$$=p+L-1+(q-p-d-L+1)/2^j.$$

Since this upper bound is satisfies when $j=1$, the process is ARFIMA(p,q_j) with $q_j\leq p+L-1+(q-p-d-L+1)/2^j$, and then (i) is true.

When $L/2-d\in\mathbb{R}_+\setminus\mathbb{Z}_+$, the term of $|1-e^{i2\pi k2^{-j}}Z^{2^{-j}}|^{L-2d}$ in the MA term is represented an infinite summation of the polynomial of $Z^{2^{-j}}$, and the above upper bound becomes infinity. Therefore (ii) is proven.

A.2.2 Proof of (iii) and (iv) of Theorem 1

For $j=1$,

$$S_{W,1}(f)=\frac{1}{2}\left[S_{\check{W},1}\left(\frac{f}{2}\right)+S_{\check{W},1}\left(\frac{f+k}{2}\right)\right],$$

So

$$
\begin{aligned}
&S_{W,1}(f)\\
&= \frac{1}{2}\left[2^{1-L}\sigma_\varepsilon^2|1-e^{i2\pi f/2}|^{L-2d}\left[\sum_{l=0}^{\frac{L}{2}-1} b_l|1+e^{i2\pi f/2}|^{2l}\right]\frac{|\Theta_q(e^{i2\pi f/2})|^2}{|\Psi_p(e^{i2\pi f/2})|^2}\right.\\
&\quad\left.+2^{1-L}\sigma_\varepsilon^2|1+e^{i2\pi f/2}|^{L-2d}\left[\sum_{l=0}^{\frac{L}{2}-1} b_l|1-e^{i2\pi f/2}|^{2l}\right]\frac{|\Theta_q(-e^{i2\pi f/2})|^2}{|\Psi_p(-e^{i2\pi f/2})|^2}\right]\\
&= \frac{1}{2}2^{1-L}\sigma_\varepsilon^2|1-e^{i2\pi f/2}|^{L-2d}|1+e^{i2\pi f/2}|^{L-2d}|\Psi_p(e^{i2\pi f/2})|^{-2}|\Psi_p(-e^{i2\pi f/2})|^{-2}\\
&\quad\times\left[|1+e^{i2\pi f/2}|^{2d-L}|\Psi_p(-e^{i2\pi f/2})|^2\left[\sum_{l=0}^{\frac{L}{2}-1} b_l|1+e^{i2\pi f/2}|^{2l}\right]|\Theta_q(e^{i2\pi f/2})|^2\right.\\
&\quad\left.+|1-e^{i2\pi f/2}|^{2d-L}|\Psi_p(e^{i2\pi f/2})|^2\left[\sum_{l=0}^{\frac{L}{2}-1} b_l|1-e^{i2\pi f/2}|^{2l}\right]|\Theta_q(-e^{i2\pi f/2})|^2\right].
\end{aligned}
$$

From Lemma 1,

$$|1-e^{i2\pi f/2}|^{L-2d}|1+e^{i2\pi f/2}|^{L-2d} = |1-e^{i2\pi f}|^{L-2d},$$

so the degree of the memory parameter of $\{W_{j,k}\}$ becomes $d-L/2$. We set this MA term as

$$|\widetilde{\Theta}_1(e^{i2\pi f})|^2 \overset{\text{set}}{=} |\widehat{\Theta}_{1,0}(e^{i2\pi f})|^2 + |\widehat{\Theta}_{1,1}(e^{i2\pi f})|^2,$$

where

$$|\widehat{\Theta}_{1,0}(e^{i2\pi f})|^2 = |1+e^{i2\pi f/2}|^{2d-L}|\Psi_p(-e^{i2\pi f/2})|^2\left[\sum_{l=0}^{\frac{L}{2}-1} b_l|1+e^{i2\pi f/2}|^{2l}\right]|\Theta_q(e^{i2\pi f/2})|^2,$$

and

$$|\widehat{\Theta}_{1,1}(e^{i2\pi f})|^2 = |1-e^{i2\pi f/2}|^{2d-L}|\Psi_p(e^{i2\pi f/2})|^2\left[\sum_{l=0}^{\frac{L}{2}-1} b_l|1-e^{i2\pi f/2}|^{2l}\right]|\Theta_q(-e^{i2\pi f/2})|^2.$$

There is no root on unit-circle which makes the above two equations zero simultaneously, and when $d-L/2\in\mathbb{N}$, the upper bound of the degree of this MA term is

$$\frac{1}{2}\left(d-\frac{L}{2}\right)+p/2+\left(\frac{L}{2}-1\right)/2+q/2 = (d+p+q-1)/2.$$

For $j \geq 2$,

$$
\begin{aligned}
&S_{W,j}(f) \\
&\quad = 2^{-j} \sum_{k=0}^{2^j-1} S_{\check{W},j}\left(\frac{f+k}{2^j}\right) \\
&\quad = 2^{-j} \prod_{k=0}^{2^j-1} \left[|1-e^{i2\pi k/2^j}e^{i2\pi f/2^j}|^{L-2d}|\Psi_p(e^{i2\pi k/2^j}e^{i2\pi f/2^j})|^{-2}\right] \\
&\qquad \times \sum_{k=0}^{2^j-1} 2^{1-L}\sigma_\varepsilon^2 \left\{\sum_{l=0}^{\frac{L}{2}-1} b_l |1+e^{i2\pi\frac{k}{2}}e^{i2\pi\frac{f}{2}}|^{2l}\right\} \\
&\qquad \times \prod_{m=0}^{j-2}\left[2^{1-L}|1+e^{i2\pi\frac{k}{2^{j-m}}}e^{i2\pi\frac{f}{2^{j-m}}}|^{2L}\sum_{l_m=0}^{\frac{L}{2}-1} b_{l_m}|1-e^{i2\pi\frac{k}{2^{j-m}}}e^{i2\pi\frac{f}{2^{j-m}}}|^{2l_m}\right] \\
&\qquad \times |\Theta_q(e^{i2\pi\frac{k}{2^j}}e^{i2\pi\frac{f}{2^j}})|^2 \prod_{h=0,h\neq k}^{2^j-1}\left[|1-e^{i2\pi h/2^j}e^{i2\pi f/2^j}|^{2d-L}|\Psi_p(e^{i2\pi h/2^j}e^{i2\pi f/2^j})|^2\right].
\end{aligned}
$$

From Lemma 1,

$$
S_{W,j}(f) \overset{\text{set}}{=} 2^{1-L}\sigma_\varepsilon^2|1-e^{i2\pi f}|^{L-2d}\frac{|\widehat{\Theta}_j(e^{i2\pi f})|^2}{|\widehat{\Psi}_j(e^{i2\pi f})|^2},
$$

where

$$
|\widehat{\Psi}_j(e^{i2\pi f})|^2 = \prod_{k=0}^{2^j-1}|\Psi_p(e^{i2\pi k/2^j}e^{i2\pi f/2^j})|^2, \quad |\widehat{\Theta}_j(e^{i2\pi f})|^2 = 2^{-j}\sum_{k=0}^{2^j-1}|\widehat{\Theta}_{j,k}(e^{i2\pi f})|^2,
$$

with

$$
\begin{aligned}
&|\widehat{\Theta}_{j,k}(e^{i2\pi f})|^2 \\
&\quad = \left\{\sum_{l=0}^{\frac{L}{2}-1} b_l |1+e^{i2\pi\frac{k}{2}}e^{i2\pi\frac{f}{2}}|^{2l}\right\} \\
&\qquad \times \prod_{m=0}^{j-2}\left[2^{1-L}|1+e^{i2\pi\frac{k}{2^{j-m}}}e^{i2\pi\frac{f}{2^{j-m}}}|^{2L}\sum_{l_m=0}^{\frac{L}{2}-1} b_{l_m}|1-e^{i2\pi\frac{k}{2^{j-m}}}e^{i2\pi\frac{f}{2^{j-m}}}|^{2l_m}\right] \\
&\qquad \times |\Theta_q(e^{i2\pi\frac{k}{2^j}}e^{i2\pi\frac{f}{2^j}})|^2 \prod_{h=0,h\neq k}^{2^j-1}\left[|1-e^{i2\pi h/2^j}e^{i2\pi f/2^j}|^{2d-L}|\Psi_p(e^{i2\pi h/2^j}e^{i2\pi f/2^j})|^2\right].
\end{aligned}
$$

There is no root on unit-circle which satisfies $|\widehat{\Theta}_{j,k}(Z)|^2 = 0$ for all $k = 0, 1, \ldots, 2^j - 1$. When

$d - L/2 \in \mathbb{N}$, the upper bound of the degree of this MA term becomes

$$\begin{aligned}\left(\frac{L}{2}-1\right)/2 + \sum_{m=0}^{j-2} 2^{-j+m}\left(\frac{3}{2}L-1\right) + q/2^j \\ +(2^j-1)\left(d-\frac{L}{2}\right)/2^j + (2^j-1)p/2^j \\ = d+p+\frac{L}{2}-1+(q-d-p-L+1)/2^j.\end{aligned}$$

This result includes the result of $j=1$, and then (iii) is proven.

When $L/2-d \in \mathbb{R}_- \setminus \mathbb{Z}_-$, $\prod_{h=0,h\neq k}^{2^j-1}|1-e^{i2\pi h/2^j}e^{i2\pi f/2^j}|^{2d-L}$ in the k-th MA term is represented as an infinite summation of the polynomial of $Z^{2^{-j}}$, and then the above upper bound becomes infinity. Therefore (iv) is true.

A.3 Proof of Corollary 1

$$\begin{aligned} S_{\check{V},j}(f) &= \mathcal{G}_{j,L}(f)S_X(f) \\ &= \prod_{m=1}^{j} 2^{1-L}|1+e^{i2\pi 2^{m-1}f}|^L \sum_{l_m=0}^{\frac{L}{2}-1} b_{l_m}|1-e^{i2\pi 2^{m-1}f}|^{2l m} S_X(f).\end{aligned}$$

Then the generalized SDF is

$$S_{V,j}(f) = \frac{1}{2^j}\sum_{k=0}^{2^j-1} S_{\check{V},j}\left(\frac{f+k}{2^j}\right). \tag{21}$$

(i) If $d \geq 0$, we can set eq.(21) as

$$S_{V,j}(f) \overset{\text{set}}{=} \prod_{k=0}^{2^j-1}|1-e^{i2\pi(f+k)/2^j}|^{-2d}\frac{|\widetilde{\Theta}_j(e^{i2\pi f})|^2}{|\widetilde{\Psi}_j(e^{i2\pi f})|^2},$$

where

$$\left|\widetilde{\Theta}_j(e^{i2\pi f})\right|^2 = \sigma_\varepsilon^2 2^{-j}\sum_{k=0}^{2^j-1}\left|\widetilde{\Theta}_{j,k}(e^{i2\pi f})\right|^2,$$

$$\begin{aligned}&\left|\widetilde{\Theta}_{j,k}(e^{i2\pi f})\right|^2 \\ &\quad = \left[\prod_{h=0,h\neq k}^{2^j-1}|1-e^{i2\pi(f+h)/2^j}|^{2d}\right]\left[\prod_{h=0,h\neq k}^{2^j-1}|\Psi_p(e^{i2\pi(f+h)/2^j})|^2\right]|\Theta_q(e^{i2\pi(f+k)/2^j})|^2 \\ &\quad\quad \times \prod_{m=1}^{j} 2^{1-L}|1+e^{i2\pi 2^{m-1}(f+k)/2^j}|^L \sum_{l_m=0}^{\frac{L}{2}-1} b_{l_m}|1-e^{i2\pi 2^{m-1}(f+k)/2^j}|^{2lm},\end{aligned}$$

and

$$|\widetilde{\Psi}_j(e^{i2\pi f})|^2 = \left|\prod_{k=0}^{2^j-1} \Psi_p(e^{i2\pi(f+k)/2^j})\right|^2.$$

From Lemma 1, the degree of the memory parameter is still d, and that of the AR term is p. When $d \in \mathbb{Z}_+$, the upper bound of the degree of the MA term is

$$\begin{aligned}(2^j-1)d/2^j + (2^j-1)p/2^j + q/2^j + \sum_{m=1}^{j} 2^{m-1-j}(L/2+L/2-1) \\ = p+d+L-1+(q-p-d-L+1)/2^j.\end{aligned}$$

When $d \in \mathbb{R}_+ \setminus \mathbb{Z}_+$, the term

$$(1-e^{i2\pi(f+h)/2^j})^d$$

becomes the infinite sum of the polynomial $e^{i2\pi f/2^j}$, so $\tilde{q} \leq \infty$.

(ii) If $d < 0$,

$$S_{V,j}(f) \overset{\text{set}}{=} \frac{|\widetilde{\Theta}_j(e^{i2\pi f})|^2}{|\widetilde{\Psi}_j(e^{i2\pi f})|^2},$$

where

$$\left|\widetilde{\Theta}_j(e^{i2\pi f})\right|^2 = \sigma_\varepsilon^2 2^{-j} \sum_{k=0}^{2^j-1} \left|\widetilde{\Theta}_{j,k}(e^{i2\pi f})\right|^2, \quad |\widetilde{\Psi}_j(e^{i2\pi f})|^2 = \left|\prod_{k=0}^{2^j-1} \Psi_p(e^{i2\pi(f+k)/2^j})\right|^2,$$

and

$$\begin{aligned}&\left|\widetilde{\Theta}_{j,k}(e^{i2\pi f})\right|^2 \\ &\quad = |1-e^{i2\pi(f+k)/2^j}|^{-2d}\left[\prod_{h=0,h\neq k}^{2^j-1} |\Psi_p(e^{i2\pi(f+h)/2^j})|^2\right] |\Theta_q(e^{i2\pi(f+k)/2^j})|^2 \\ &\quad\quad \times \prod_{m=1}^{j} 2^{1-L}|1+e^{i2\pi 2^{m-1}(f+k)/2^j}|^L \sum_{l_m=0}^{\frac{L}{2}-1} b_{l_m}|1-e^{i2\pi 2^{m-1}(f+k)/2^j}|^{2lm}.\end{aligned}$$

Since

$$|1-e^{i2\pi(f+k)/2^j}|^{-2d}\prod_{m=1}^{j}|1+e^{i2\pi 2^{m-1}(f+k)/2^j}|^{-2d} = |1-e^{i2\pi f}|^{-2d},$$

the above equation becomes

$$\begin{aligned}
&\left|\widetilde{\Theta}_{j,k}(e^{i2\pi f})\right|^2 \\
&\quad = |1-e^{i2\pi f}|^{-2d}\left[\prod_{h=0,h\neq k}^{2^j-1}|\Psi_p(e^{i2\pi(f+h)/2^j})|^2\right]|\Theta_q(e^{i2\pi(f+k)/2^j})|^2 \\
&\qquad \times\prod_{m=1}^{j}2^{1-L}|1+e^{i2\pi 2^{m-1}(f+k)/2^j}|^{L+2d}\sum_{l_m=0}^{\frac{L}{2}-1}b_{l_m}|1-e^{i2\pi 2^{m-1}(f+k)/2^j}|^{2lm}.
\end{aligned}$$

So the negative memory d survives, and the rest terms becomes the k-th MA term. When $d \in \mathbb{Z}_-$, the upper bound of the degree of the MA term is

$$\begin{aligned}
&(2^j-1)p/2^j + q/2^j + \sum_{m=1}^{j}2^{m-1-j}(L/2+d+L/2-1) \\
&\quad = p+d+L-1+(q-p-d-L+1)/2^j.
\end{aligned}$$

When $d \in \mathbb{R}_- \setminus \mathbb{Z}_-$, the term that

$$(1+e^{i2\pi 2^{m-1}(f+k)/2^j})^{L/2+d}$$

becomes the infinite sum of the polynomial $e^{i2\pi 2^{m-1}f/2^j}$.

In the sequel, from (i) and (ii), Corollary 1 is true. □

A.4 Proof of Corollary 2

Since $S_X(f) = \sigma_X^2$,

$$S_{W,j}(f) = 2^{-j}\sigma_X^2\sum_{k=0}^{2^j-1}\mathcal{H}_{j,L}\left(\frac{f+k}{2^j}\right), \quad S_{V,j}(f) = 2^{-j}\sigma_X^2\sum_{k=0}^{2^j-1}\mathcal{G}_{j,L}\left(\frac{f+k}{2^j}\right).$$

From $\mathcal{H}_{j,L}(f) = |H_{j,L}(f)|^2 = |\sum_{l=0}^{L_j-1}h_{j,l}e^{i2\pi fl}|^2$,

$$2^{-j}\sum_{k=0}^{2^j-1}\mathcal{H}_{j,L}\left(\frac{f+k}{2^j}\right) = 2^{-j}\sum_{k=0}^{2^j-1}\sum_{l=0}^{L_j-1}h_{j,l}^2 + 2^{-j}\sum_{l\neq n}h_{j,l}h_{j,n}\sum_{k=0}^{2^j-1}e^{i2\pi\frac{f+k}{2^j}(l-n)}.$$

Because $\sum_{l=0}h_{j,l}^2 = 1$ and $\sum_{k=0}^{2^j-1}e^{i2\pi\frac{f+k}{2^j}(l-n)} = 0$ for $l \neq n$,

$$S_{W,j}(f) = \sigma_X^2.$$

From the similar argument and $\sum_{l=0}^{L_j-1}g_{j,l}^2 = 1$, $S_{V,j}(f) = \sigma_X^2$. So, eq.(18) is true. □