Realized Factor GARCH:

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EUI & CREATES

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Realized Factor GARCH

- Multivariate GARCH model that utilizes realized measures of volatility and covolatility
- Hierarchical Structure
  - Market Returns + Realized Measure (RM)
    Modeled with Realized EGARCH.
  - K Sector Returns + RMs
    Modeled jointly, conditional on market.
  - Individual Assets + RMs
    modeled conditional on $K+1$ Market and Sector returns

Realized GARCH Framework

- Enhance GARCH models to include realized measures of volatility.
- E.g. Realized Variance $x_t = \sum_{i=1}^{n} y_{i,t}^2$ in place of squared returns

$$h_t = \omega + \beta h_{t-1} + \gamma x_{t-1}.$$

- where $y_{i,t}$ are intraday high-frequency data.

Figure with HF DATA

- Figure

Realized GARCH Framework

- Key for Realized GARCH: **Measurement Equation** (Takahashi, Omori, and Watanabe, 2009)

$$x_t = \xi + \varphi h_t + \mathrm{error}_t.$$

- Similar for covariance, and correlation, ρ_t, i.e.

$$f(y_t) = \tilde{\xi} + \tilde{\varphi} f(\rho_t) + \mathrm{error}_t,$$

some function f, with y_t a realized correlation:

$$y_t = \frac{x_{ij,t}}{\sqrt{x_{ii,t} x_{jj,t}}}.$$

Realized GARCH... So Far..

- Hansen, Huang & Shek (JoAE, 2012):
  Univariate + Single Realized Measure.
- Hansen & Huang (WP, 2012):
  Univariate + Multiple Realized Measures + Refinements
- Hansen, Lunde & Voev (WP, 2012)
  Multivariate: Market + Conditional Model for Each Individual Asset
  "single factor" evidence of residual un-modeled correlation.

Realized Beta-GARCH

Table 2: Unconditional correlations $\hat{z}_{i,t}$. (Grouped by GICS)

	Energy	Materials	Industrials	Consumer Discretionary	Consumer Staples	Healthcare	Financials	Information Technology	Telecommun. Services	Utilities
Energy	0.599	0.324	0.275	0.202	0.160	0.180	0.229	0.217	0.197	0.298
Materials		0.418	0.377	0.309	0.243	0.244	0.342	0.299	0.273	0.285
Industrials			0.402	0.331	0.261	0.264	0.357	0.323	0.288	0.286
Consumer Discretionary				0.323	0.238	0.234	0.327	0.283	0.252	0.238
Consumer Staples					0.266	0.211	0.263	0.207	0.216	0.235
Healthcare						0.263	0.260	0.231	0.218	0.223
Financials							0.429	0.302	0.290	0.300
Information Technology								0.360	0.268	0.230
Telecommun. Services									0.368	0.250
Utilities										0.487

Some ~~Unfinished Business~~ Unexplained Correlation

Table 3: Unconditional correlations $\hat{w}_{i,t}$. (Grouped by GICS)

	Energy	Materials	Industrials	Consumer Discretionary	Consumer Staples	Healthcare	Financials	Information Technology	Telecommun. Services	Utilities
Energy	0.484	0.092	0.015	-0.036	-0.044	-0.019	-0.057	-0.023	-0.025	0.099
Materials		0.160	0.082	0.043	0.007	0.004	0.023	0.018	0.006	0.021
Industrials			0.105	0.059	0.021	0.021	0.026	0.037	0.013	0.013
Consumer Discretionary				0.098	0.032	0.020	0.049	0.033	0.010	-0.005
Consumer Staples					0.106	0.036	0.021	-0.015	0.017	0.041
Healthcare						0.097	0.012	0.010	0.011	0.022
Financials							0.145	0.002	0.011	0.025
Information Technology								0.131	0.022	-0.023
Telecommun. Services									0.185	0.028
Utilities										0.341

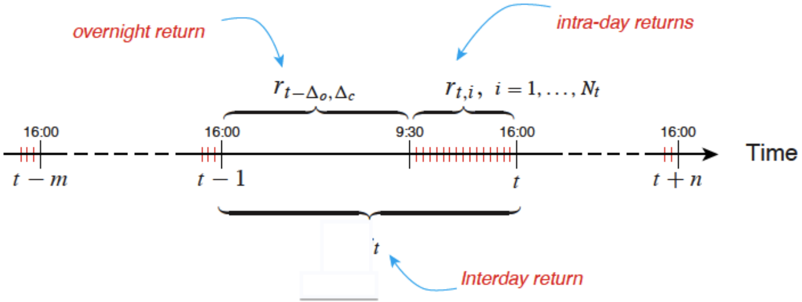
Realized Factor GARCH

- A Core Model
  - For the Market Return (SPY)
  - Ten Sectors (XLB, XLV, XLP, XLY, XLE, XLF, XLI, XLK, XLU, ~~VOX~~)
- A Separate (Factor) Model for Each Individual Asset
  - Conditional on CORE Model
  - Dynamic Multifactor structure.

Core Model (Market)

- Market Variables:
- R_{0t} Market return
  (SPX close-to-close returns used as proxy)
- X_{0t} Realized measure of market volatility
  (Realized Kernel from intraday returns)

High-Frequency Data and Daily Returns



Core Model (Market)

GARCH-X Component

- Define

$$H_{0t} = \mathrm{var}(R_{0t}|\mathcal{F}_{t-1}) \qquad \text{and} \qquad M_{0t} = \mathrm{E}(R_{0t}|\mathcal{F}_{t-1}).$$

  Assume constant mean $M_0 = M_{0t}$.

- Studentized Return

$$Z_{0t} = \frac{R_{0t} - M_0}{\sqrt{H_{0t}}}.$$

- GARCH Equation

$$\log H_{0t} = A_0 + B_0 \log H_{0,t-1} + C_0 \log X_{0,t-1} + \tau^{(0)}(Z_{0,t-1})$$

- $\tau^{(0)}(z) = \tau_1 z + \tau_2(z^2 - 1)$ is a leverage function.

- The Realized Measure $\log X_{0t}$ is a noisy (possibly biased) measure of $\log H_{0t}$.
- Measurement Equation

$$\log X_{0,t} = F_0 + \log H_{0,t} + \delta^{(0)}(Z_{0,t}) + U_{0,t},$$

- where $\delta^{(0)}$ is another leverage function.

Core Model (Market)

- Realized EGARCH Model:

$$
\begin{aligned}
R_{0t} &= M_0 + \sqrt{H_{0t}} Z_{0t} \\
\log H_{0t} &= A_0 + B_0 \log H_{0,t-1} + C_0 \log X_{0,t-1} + \tau^{(0)}(Z_{0,t-1}) \\
\log X_{0,t} &= F_0 + \log H_{0,t} + \delta^{(0)}(Z_{0,t}) + U_{0,t},
\end{aligned}
$$

Return Equation,
GARCH Equation, &
Measurement Equation.

Core Model Variables

- Core Model Variables:
  - R_{0t} Market return (SPX returns used as proxy)
  - R_{jt}, $j = 1, \ldots, K$ Sector j Returns (Energy, Financials, etc.)
  - X_{0t} Realized measure of market volatility
  - X_{jt} Realized measure of Sector j volatility.
  - Y_{jt} realized correlation measure between R_{0t} and R_{jt}.

$$X_{jt} = \frac{X_{0jt}}{\sqrt{X_{0t} X_{jt}}},$$

where X_{0jt} is a realized covariance measure.

Core Model (Sector Returns)

- Similar Structure

$$\begin{aligned} R_{jt} &= M_j + \sqrt{H_{jt}} Z_{jt}, \\ \log H_{jt} &= A_j + B_j \log H_{j,t-1} + C_j \log X_{j,t-1} + D_j \log H_{0t} + \tau^{(j)}(Z_{j,t-1}), \\ \log X_{jt} &= F_j + \log H_{jt} + \delta^{(j)}(Z_{jt}) + U_{jt}, \end{aligned}$$

where we allow for direct impact from H_{0t} to H_{jt} .

Core Model (Sector Returns)

Connecting Sector Returns with Market Returns

- Modeling of dynamic correlations

$$\rho_{jt} = \mathrm{corr}(R_{0t}, R_{jt} | \mathcal{F}_{t-1}).$$

- GARCH and Measurement Equation

$$\begin{aligned} F(\rho_{j,t}) &= \Xi_j + \Phi_j F(\rho_{jt-1}) + \Gamma_j F(Y_{j,t-1}), \\ F(Y_{j,t}) &= \Psi_j + F(\rho_{jt}) + V_{jt}, \end{aligned}$$

where $F(\cdot)$ denote the Fisher transform.

Empirical Results (Core Model)

Correlation between Sector Returns & Market Returns

- Plot: ρ_{jt}.

- Since $\rho_{jt} = \mathrm{cov}(Z_{0t}, Z_{jt})$, we have..
- ...for $Z_t = (Z_{0t}, \ldots, Z_{Kt})'$ that

$$\mathrm{var}(Z_t) = \begin{pmatrix} 1 & \bullet & \bullet & \bullet & \bullet \\ \rho_{1t} & 1 & \bullet & \bullet & \bullet \\ \rho_{2t} & ? & 1 & \bullet & \bullet \\ \vdots & \vdots & \ddots & \ddots & \bullet \\ \rho_{1t} & ? & \cdots & ? & 1 \end{pmatrix}.$$

Core Model (Sector Returns)

Residual Sector Returns Interdependence

- Define conditional and studentized sector returns

$$W_{jt} = \frac{Z_{jt} - \rho_{jt} Z_{0t}}{\sqrt{1 - \rho_{jt}^2},}.$$

- Assume constant correlation

$$\omega_{ij} = \mathrm{cov}(W_{ij}, W_{jt}).$$

- With $W_t = (W_{1t}, \ldots, W_{Kt})'$, let

$$\Omega = \mathrm{var}(W_t).$$

Empirical Results: Residual Sector Correlation

Preliminary Estimate

$$\hat{\Omega} =$$

	XLB	XLV	XLP	XLY	XLE	XLF	XLI	XLK	XL
XLB	1.010								
XLV	-0.099	1.000							
XLP	-0.084	0.215	0.990						
XLY	0.010	0.014	0.149	0.985					
XLE	0.283	-0.140	-0.164	-0.248	1.021				
XLF	-0.091	-0.021	0.011	0.153	-0.294	1.011			
XLI	0.256	-0.008	0.038	0.241	-0.063	0.009	1.005		
XLK	-0.038	-0.125	-0.070	0.111	-0.198	-0.132	0.076	0.999	
XLU	0.051	0.046	0.093	-0.090	0.196	-0.049	-0.063	-0.142	1.0

- Let

$$\rho_t = \begin{pmatrix} \rho_{1t} \\ \vdots \\ \rho_{Kt} \end{pmatrix} \quad \text{and} \quad \Lambda_t = \begin{pmatrix} \sqrt{1-\rho_{1t}^2} & & 0 \\ & \ddots & \\ 0 & & \sqrt{1-\rho_{Kt}^2} \end{pmatrix}.$$

- From definition of W_{jt},

$$Z_{jt} = \rho_{jt} Z_{0t} + \sqrt{1-\rho_{jt}^2}\, W_{jt}.$$

- In compact form:

$$Z_t = \begin{bmatrix} 1 & 0 \\ \rho_t & \Lambda_t \end{bmatrix} \begin{pmatrix} Z_{0t} \\ W_t \end{pmatrix}.$$

Core Model (Sector Returns)

Residual Sector Returns Interdependence

- With

$$Z_t = \begin{bmatrix} 1 & 0 \\ \rho_t & \Lambda_t \end{bmatrix} \begin{pmatrix} Z_{0t} \\ W_t \end{pmatrix}.$$

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$$\begin{aligned} \Sigma_{Z_t} = \mathrm{var}(Z_t) &= \begin{bmatrix} 1 & 0 \\ \rho_t & \Lambda_t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} \begin{bmatrix} 1 & \rho_t' \\ 0 & \Lambda_t \end{bmatrix}. \\ &= \begin{bmatrix} 1 & \rho_t' \\ \rho_t & \rho_t \rho_t' + \Lambda_t \Omega \Lambda_t \end{bmatrix} \end{aligned}$$

Core Model (Sector Returns)

Key Core Quantity

- The inverse of $\Sigma_{Z_t} = \begin{bmatrix} 1 & \rho_t' \\ \rho_t & \rho_t\rho_t' + \Lambda_t\Omega\Lambda_t \end{bmatrix}$, is central in factor models for all individual assets.
-

$$\begin{aligned} \Sigma_{Z_t}^{-1} &= \begin{bmatrix} 1 & \rho_t' \\ 0 & \Lambda_t \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \Omega^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \rho_t & \Lambda_t \end{bmatrix}^{-1}. \\ &= \begin{bmatrix} 1 & -\rho_t'\Lambda_t^{-1} \\ 0 & \Lambda_t^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \Omega^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\Lambda_t^{-1}\rho_t & \Lambda_t^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 1 + \rho_t'\Lambda_t^{-1}\Omega^{-1}\Lambda_t^{-1}\rho_t & -\rho_t'\Lambda_t^{-1}\Omega^{-1}\Lambda_t^{-1} \\ -\rho_t\Lambda_t^{-1}\Omega^{-1}\Lambda_t^{-1} & \Lambda_t^{-1}\Omega^{-1}\Lambda_t^{-1} \end{bmatrix} \end{aligned}$$

Core Model (Sector Returns)

Key Core Quantity

- Plot of $\Sigma_{Z_t}^{-1}$ is some way?

Factor Model (Individual Return)

Basic Model

- We start simple

$$\begin{aligned} r_{it} &= m_i + \sqrt{h_{it}} z_{it}, \\ \log h_{it} &= a_i + b_i \log h_{i,t-1} + c_i \log x_{i,t-1} + \tau^{(i)}(z_{i,t-1}), \\ \log x_{it} &= f_i + \log h_{it} + \delta^{(i)}(z_{i,t}) + u_{it}, \end{aligned}$$

- Related to Core model through correlation with Z_t and U_t.

- Let

$$\lambda_i = \mathrm{cov}(Z_t, z_{it}) \in \mathbb{R}^{K+1}$$

- So that

$$z_{it}|Z_t \sim N(\lambda_i' \Sigma_{Z_t}^{-1} Z_t, 1 - \lambda_i' \Sigma_{Z_t}^{-1} \lambda_i).$$

- Similarly

$$u_{it}|U_t \sim N(\Sigma_{iU} \Sigma_{UU}^{-1} U_t, 1 - \Sigma_{iU} \Sigma_{UU}^{-1} \Sigma_{Ui}).$$

- List $\lambda_i = \mathrm{cov}(Z_t, z_{it}) \in \mathbb{R}^{K+1}$ for some individual assets

- List $\tilde{\lambda}_i = \mathrm{cov}(\tilde{W}_t, z_{it}) \in \mathbb{R}^{K+1}$ for some individual assets, where $\tilde{W}_i = (Z_{0t}, W_t')'$.
- Better to keep model $\tilde{\lambda}_i$ as constant?

Factor Model (Individual Return)

Factor Loadings

- Factor loadings for i-th asset, are given by

$$\beta_{it} = \mathrm{var}(R_t)^{-1}\mathrm{cov}(R_t, r_{it})$$

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$$\begin{aligned} \beta_{it} &= D_{H_t}^{-1/2}\mathrm{var}(Z_t)^{-1}\mathrm{cov}(Z_t, z_{it})\sqrt{h_{it}} \\ &= D_{H_t}^{-1/2}\Sigma_{Z_t}^{-1}\lambda_i\sqrt{h_{it}} \end{aligned}$$

where $D_{H_t} = \mathrm{diag}(H_{0t}, H_{1t}, \ldots, H_{Kt})$.

- $\beta_{it} = D_{H_t}^{-1/2} \Sigma_{Z_t}^{-1} \lambda_i \sqrt{h_{it}}$.
- Factor loadings for studentized returns $Z_t \mapsto z_{it}$:
- $$\tilde{\beta}_{it} = \Sigma_{Z_t}^{-1} \lambda_i$$
- Factor loadings: Studentized orthogonalized basis $(Z_{0t}, W_t) \mapsto z_{it}$
- $$\beta_{it}^{\perp} = \begin{bmatrix} 1 & 0 \\ \rho_t & \Lambda_t \end{bmatrix} \Sigma_{Z_t}^{-1} \lambda_i$$

Empirical Results

Factor Loadings for MSFT

- PLOT β_{it}.

- PLOT $\tilde{\beta}_{it} = \Sigma_{Z_t}^{-1} \lambda_i$

- PLOT $\beta_{it}^{\perp}$ (perhaps excluding first element).

Conclusion

- Realized Factor GARCH
  - **Multivariate GARCH Model with Realized Measures**
  - Flexible Factor Structure
- Key Features
  - Simple **Parsimonious** Structure
  - **Easy** to estimate.
  - Easy to Scale to Vast Systems
- Empirical Results
  - Intuitive results
  - ... much more to come.