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On the Realizability of Social Preferences in Three-Party Parliamentary Democracies

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On the realizability of social preferences in three-party parliamentary democracies*

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Abstract

It is widely held that compared with a legislature with a single majority party, a multi-party legislature achieves more precise representation of society. But the scope of such an advantage that a multi-party system has is rarely discussed. We study the range of social preferences that a three-party system can realize through majority voting. We present a procedure to construct a three-party system that will induce the policy choice specified by a given social preference relation. We provide a sufficient condition for a social preference relation to be compatible with some three-party system. The condition describes a certain restriction on the structure of cycles of social preference relations.

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1 Introduction

A widely held view about parliamentary systems is that while a single majority party makes decisions more quickly, a multi-party system achieves more precise representation of society. But the scope of such an advantage that a multi-party system has is rarely discussed.

In this paper we limit the notion of representation of society to the sense that a legislature realizes a given social preference relation over alternative policies. A social preference relation is possibly intransitive. For example, it may represent cyclic social majority preferences as illustrated by the well-known "Paradox of Voting." Thus a legislature that perfectly represents the society may induce cyclic decisions when it deals with multiple agenda.

We focus on legislatures with at most three parties where policy is chosen through majority voting. Each party is assumed to have transitive preferences. We study the range of social preferences such a legislature can realize. By limiting attention to three-party systems, we try to understand the minimal effect arising from the absence of a single-party majority. As we will see below, it has been shown that without any constraint on the number of parties, representation of society is always possible. But it seems unrealistic to expect that arbitrarily many parties may form.[1]

We present a theoretical procedure to construct a three-party legislature in such a way that it realizes a given social preference relation. The procedure, which we call "labeling," designates a way in which, through parallel elimination of cycles of the social preference relation, we may arrive at a desired profile of parties' preferences. It is possible, however, that some social preference relation cannot be induced by any collection of three parties. We provide a sufficient condition for a social preference relation to be compatible with some three-party system. The condition describes a certain restriction on the structure of cycles of the social preferences.

The topic of this paper is closely related to the well-known Paradox of Voting. The Paradox states that three or more voters may induce cyclic majority preferences over a set of alternatives even when each voter has transitive preferences. Thus the social majority preference relation may be intransitive.[2] An intransitive

[1]Lijphart (1994) reports the effective numbers of parties for 70 electoral systems in 27 countries between 1945 and 1990. The average effective number of parliamentary parties over these systems is 3.4.

[2]DeMeyer and Plott (1970), Gehrlein and Fishburn (1975), and Jones, Radcliff, Taber, and Timpone (1995) compute the probability that majority voting by a random set of voters produces

social preference relation cannot be realized if the legislature consists of at most two parties with transitive preferences (except under a special tie-breaking rule). Three is therefore the minimum number of parties that can realize an intransitive social preference relation. The main question we explore is which among those cyclic preferences are indeed generated by some three-party systems.

McGarvey (1953) demonstrates that the Paradox of Voting can be formulated in a more general way. He shows that any preference relation over n alternatives is generated by some set of voters with transitive preferences. Stearns (1959) provides a lower bound for the minimum number of voters necessary to induce all preference relations over n alternatives. The lower bound is of the form $c_1 n/\log n$, with some constant c_1. Erdös and Moser (1964) provide a sharp upper bound of the form $c_2 n/\log n$.

These results imply that if we wish to count all possible social majority preference relations as potential social preference relations (and if the population of society is large enough relative to the number of alternatives), we should admit all preference relations. This is why we put no a priori restriction on the social preferences.

From the above estimates by Stearns and Erdös and Moser it also follows that for any upper bound k on the number of parties and for a sufficiently large number of policies, there exist some preference relation that cannot be realized by any set of k or less parties. A more recent study by Alon, Brightwell, Kierstead, Kostochka, and Winkler (2006) provides a condition which is useful in verifying that a given social preference relation is incompatible with any three-party system. A "dominating set" is defined as a subset A of alternatives such that for any alternative x, some alternative $a \in A$ is majority-preferred to x. Alon et al. show that any three-voter majority preference relation has a dominating set containing at most three alternatives. For our present setting, this property is naturally interpreted as a necessary condition for a social preference relation to be realized by some three-party system. In this paper, we rather focus on deriving sufficient conditions.

The rest of the paper is organized as follows. In Section 2 we define a framework to analyze three-party legislatures. In Section 3 we propose a simple method called "labeling" to construct a three-party system which realizes a given social preference relation. It turns out, however, that while any such three-party system

a cycle. The probability depends on the numbers of voters and alternatives. Jones et al. (1995) estimate that the probability of a majority cycle increases with the number n of alternatives, and is already very close to 1 when $n = 10$.

is necessarily the outcome of a labeling, some labeling fails to produce such a three-party system. In Section 4, we impose an additional condition on the labeling procedure, called "orientedness," and show that any oriented labeling produces a desired profile of parties. In Section 5, we provide a sufficient condition for a social preference relation to admit an oriented labeling, and hence to be realized by some three-party system. Finally, in Section 6 we conclude.

2 The model

Consider a society in which policy is chosen through majority voting in a legislature. The society has a collective preference relation over the set of policies. It seeks to send a set of representatives to the legislature in such a way that they will realize policy choices in accordance with the social preferences. Suppose that the society can only choose a set of representatives that is partitioned into at most three parties.

Let X be a finite set of *policies*.

A *social preference relation* is a complete and asymmetric preference relation $\succ_S$ on X. An example is the majority preference relation of the society.

A *3-party system* is a triple $\succ_P = (\succ_1, \succ_2, \succ_3)$ in which each *party i* has a complete, asymmetric, and transitive preference relation $\succ_i$ over X. A 3-party system represents a possible composition of a legislature, where the parties together occupy all seats, and no single party has a majority of seats. (Thus a 3-party system can be interpreted simply as a profile of three voters each having one vote.)

We denote by $M(\succ_P)$ the majority preference relation generated by the 3-party system $\succ_P$: for any pair $\{x, y\} \subset X$ of distinct policies,[3] $xM(\succ_P)y$ if and only if $x \succ_i y$ for at least two parties $i \in \{1, 2, 3\}$.

A 3-party system $\succ_P$ is said to *represent* a social preference relation $\succ_S$ if $M(\succ_P) = \succ_S$. A social preference relation $\succ_S$ is called *3-party representable* if some 3-party system represents $\succ_S$.

Example 1. (*Voting Paradox*) Let $X = \{a, b, c\}$, and consider a cyclic social preference relation $a \succ_S b \succ_S c \succ_S a$. Define a 3-party system $\succ_P$ as follows: $a \succ_1 b \succ_1 c$ and $a \succ_1 c$; $b \succ_2 c \succ_2 a$ and $b \succ_2 a$; and $c \succ_3 a \succ_3 b$ and $c \succ_3 b$. Then $\succ_P$ represents $\succ_S$. □

[3] Throughout the paper, an expression of the form "$\{x, y\}$" always represents an unordered pair of distinct elements.

Remark 1. A legislature with a single-party majority (e.g., a 2-party system without majority ties) can be expressed as a 3-party system. For example, a 2-party system $(\succ_1, \succ_2)$ where party 1 is a majority in the legislature is defined as the 3-party system $(\succ_1, \succ_1, \succ_2)$. □

3 3-party representation and labeling

As illustrated in Example 1, for a 3-party system to represent cyclic social preferences, each party must disagree with the society on some policy pair. In this section, we show that if a 3-party system represents a social preference relation, then the overall pattern of such disagreement must be a "labeling" — a rule that assigns party names to policy pairs subject to a certain condition. Consequently, a necessary condition for a social preference to be 3-party representable is that it admits a labeling. One might expect, conversely, that labeling always produces a 3-party system that represents a given social preference relation. But it turns out that this is not true.

Given a social preference relation $\succ_S$ and a party i's preference relation $\succ_i$, the *disagreement set* $D(\succ_i, \succ_S)$ is defined as the set of policy pairs on which the two preferences disagree:

$$D(\succ_i, \succ_S) = \{\{x, y\} \subset X : (x \succ_i y \text{ and } y \succ_S x) \text{ or } (y \succ_i x \text{ and } x \succ_S y)\}.$$

Given a pair $(\succ_P, \succ_S)$ of a 3-party system and a social preference relation, let $D(\succ_P, \succ_S)$ denote the family of disagreement sets: $D(\succ_P, \succ_S) = (D(\succ_i, \succ_S))_{i=1,2,3}$.

A *labeling* of a social preference relation $\succ_S$ is a family of three subsets of policy pairs $L = (L_i)_{i=1,2,3}$ (where we say "a policy pair $\{x, y\}$ has label i" if $\{x, y\} \in L_i$) such that:

(L1) each policy pair has at most one label; and

(L2) the three policy pairs in each 3-cycle of $\succ_S$ have distinct labels.

Figures 1 and 2 illustrate examples of labeling, where "$a \to b$" indicates the relation $a \succ_S b$.

Note that a labeling L of a social preference relation $\succ_S$ determines a unique triple $\succ = (\succ_i)_{i=1,2,3}$ of (not necessarily transitive) preferences, via $D(\succ, \succ_S) = L$. Thus each label i may be seen as specifying policy pairs on which "party i" (with a possibly intransitive preference) disagrees with the social preferences. From this view, the condition (L1) ensures that at least two parties agree with the society on each policy pair. The condition (L2), then, says that each party disagrees with the society on exactly one policy pair in each 3-cycle of the social preferences.

Proposition 1. *If a 3-party system* $\succ_P$ *represents a social preference relation* $\succ_S$*, then the family of disagreement sets* $D(\succ_P, \succ_S)$ *is a labeling of* $\succ_S$*. Thus, if a social preference relation* $\succ_S$ *is 3-party representable, then* $\succ_S$ *has a labeling.*

Proof. Since $\succ_P$ represents $\succ_S$, for each policy pair $\{x, y\}$, at most one party can disagree with the society on $\{x, y\}$. Thus (L1) holds. For each party i, since $\succ_i$ is transitive, $\succ_i$ must disagree with $\succ_S$ on at least one policy pair contained in each 3-cycle of $\succ_S$. Thus for each 3-cycle C, the map that assigns to each policy pair $\{x, y\}$ in C the party that disagrees with $\succ_S$ on $\{x, y\}$ must be one-to-one. Thus (L2) holds. □

Remark 2. The converse of Proposition 1 does not hold: for some social preference relation $\succ_S$ and some labeling L of $\succ_S$, the triple $\succ = (\succ_i)_{i=1,2,3}$ defined by $D(\succ, \succ_S) = L$ is not a 3-party system satisfying the transitivity assumption. Consider the labeling illustrated in Figure 2. Then for the triple $\succ$ defined as above, (a, c, e, a) is a cycle of $\succ_1$ (and (a, e, c, a) is a cycle of $\succ_2$). This example indicates the fact that while a labeling is designed so that each label eliminates cycles that exists in the social preferences, it may give rise to a new preference cycle. □

Remark 3. We do not know how strong the condition that a social preference relation admits a labeling is. Reformulating a labeling in the language of graph theory might help examine this question.

Define a simple graph $G(\succ_S) = (V, E)$ as follows: the vertex set V is defined as the set of all policy pairs belonging to some 3-cycles of $\succ_S$; and the edge set E is defined so that two policy pairs $\{x, y\}$ and $\{v, w\}$ in V are adjacent if they belong to a common 3-cycle of $\succ_S$. Figure 6 illustrates a typical form of graph G.

Up to the labels assigned to policy pairs that do not belong to any 3-cycle, a labeling of $\succ_S$ is equivalently described as a *3-coloring* of the graph $G(\succ_S)$ (i.e., coloring verteces of the graph in three colors so that any two adjacent verteces have different colors). Thus, a social preference $\succ_S$ is 3-party representable only if the graph $G(\succ_S)$ is 3-colorable. However, we have not obtained a useful necessary condition for 3-colorability of $G(\succ_S)$.[4] □

[4]A result obtained by Borodin, Glebov, Montassier, and Raspaud (2009) implies that a sufficient condition for $G(\succ_S)$ to be 3-colorable is that $G(\succ_S)$ is “planar” (i.e., can be drawn on the plane without intersecting edges) and $G(\succ_S)$ contains no “cycles” (i.e., no closed paths) of lengths 5 and 7.

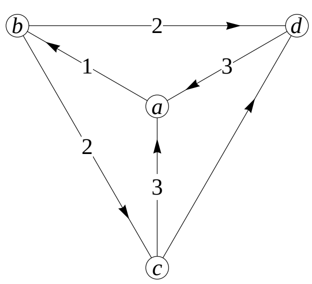

Figure 1: Oriented labeling

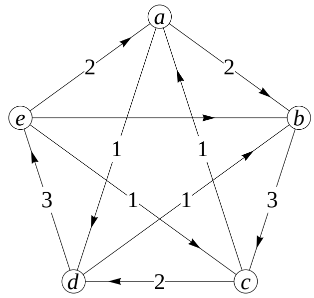

Figure 2: Non-oriented labeling

4 Oriented labeling

Remark 2 says that while any 3-party system representing a social preference relation can be constructed as the outcome of a labeling, some labeling fails to produce a 3-party system satisfying the transitivity condition on the parties' preferences. We saw that this occurs because when the social ordering on a policy pair is reversed according to the labeling, a new cycle may arise. In this section, we show that such effects will offset each other if we impose an additional condition on the labeling, which we call "orientedness." An oriented labeling successfully produces a 3-party system with transitive party preferences.

A labeling L of a social preference relation $\succ_S$ is called *oriented* if:

(O1) whenever both (a, b, c, a) and (a, b, d, a) are 3-cycles of $\succ_S$, policy pairs $\{b, c\}$ and $\{b, d\}$ (and hence pairs $\{c, a\}$ and $\{d, a\}$) have the same label; and

(O2) a policy pair has a label only if it belongs to some 3-cycle of $\succ_S$.

The labeling in Figure 1 is oriented, but the labeling in Figure 2 is not oriented.

As before, we interpret a labeling L as determining, through the equation $D(\succ_P, \succ_S) = L$, a 3-party system $\succ_P$ in which some party may have an intransitive preference relation. Recall that in the definition of a labeling, (L1) requires that each party disagrees with the society on exactly one policy pair in each 3-cycle of the social preferences. The condition (O1) says that, moreover, the association of such anti-social parties must be ordered in the same way for any two adjacent 3-cycles. The condition (O2) just says that the legislature must unanimously agree

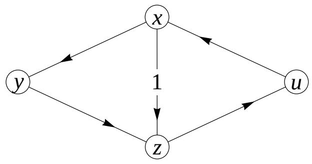

Figure 3: Case (a)

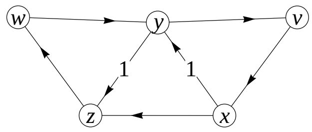

Figure 4: Case (b)

with the society on any policy pair which does not belong to any 3-cycle of the social preferences.

Proposition 2. *If L is an oriented labeling of a social preference relation $\succ_S$, then the triple $\succ = (\succ_i)_{i=1,2,3}$ determined by $D(\succ, \succ_S) = L$ is a 3-party system which represents $\succ_S$. Thus, if a social preference relation $\succ_S$ has an oriented labeling, then it is 3-party representable.*

Proof. By (L1), for any policy pair $\{x, y\}$, there are at least two i such that $\succ_i$ and $\succ_S$ agree on $\{x, y\}$.

Thus it remains to show that each party's preference relation is transitive. It suffices to check that $\succ_1$ is transitive. Suppose on the contrary that $\succ_1$ has a 3-cycle in three policies $\{x, y, z\}$.

The three policies $\{x, y, z\}$ cannot constitute a cycle of $\succ_S$: if (x, y, z, x) or (x, z, y, x) is a 3-cycle of $\succ_S$, by (L2), $\succ_1$ and $\succ_S$ disagree on exactly one policy pair in $\{x, y, z\}$ so that $\succ_1$ is transitive on $\{x, y, z\}$.

Thus suppose without loss of generality that

$$x \succ_S y \succ_S z \text{ and } x \succ_S z.$$

There are two possibilities:

(a) pair $\{x, z\}$ is labeled 1, but pairs $\{x, y\}$ and $\{y, z\}$ are not; or

(b) pairs $\{x, y\}$ and $\{y, z\}$ are labeled 1, but pair $\{x, z\}$ is not.

See Figures 3 and 4.

Case (a). Because pair $\{x, z\}$ is labeled 1, by (O2), there must be an alternative $u \notin \{x, y, z\}$ such that $C_u = (x, z, u, x)$ is a 3-cycle of $\succ_S$.

Suppose $y \succ_S u$. Then (x, y, u, x) is also a 3-cycle of $\succ_S$. Since pair $\{x, z\}$ is labeled 1 in 3-cycle $C_u = (x, z, u, x)$, by (O1), pair $\{x, y\}$ must be labeled 1, a contradiction.

Now suppose $u \succ_S y$. Then (y, z, u, y) is a 3-cycle of $\succ_S$. Since pair $\{x, z\}$ is labeled 1 in 3-cycle $C_u = (x, z, u, x)$, by (O1), pair $\{y, z\}$ must be labeled 1, a contradiction.

Thus, case (a) is impossible.

Case (b). Since pairs $\{x, y\}$ and $\{y, z\}$ are labeled 1, by (O2), there must be two alternatives $v, w \notin \{x, y, z\}$ such that $C_v = (x, y, v, x)$ and $C_w = (y, z, w, y)$ are 3-cycles of $\succ_S$. In particular, $y \succ_S v$, $w \succ_S y$, and hence $v \neq w$.

Suppose $v \succ_S w$. Then (w, y, v, w) is a 3-cycle of $\succ_S$. Since pair $\{x, y\}$ is labeled 1 in 3-cycle $C_v = (x, y, v, x)$, by (O1), pair $\{w, y\}$ must be labeled 1. This is impossible, because it implies that 3-cycle $C_w = (y, z, w, y)$ contains two pairs labeled 1, which violates (L2).

Now suppose $w \succ_S v$.

First consider the case where $w \succ_S x$. In this case, (x, z, w, x) is a 3-cycle of $\succ_S$. Since pair $\{y, z\}$ is labeled 1 in 3-cycle $C_w = (y, z, w, y)$, by (O1), pair $\{x, z\}$ must be labeled 1, which contradicts (b).

Now suppose $x \succ_S w$. Then (x, w, v, x) is a 3-cycle of $\succ_S$. Then since $C_v = (x, y, v, x)$ is a 3-cycle of $\succ_S$, by (O1), pair $\{w, x\}$ must be labeled 1. Then, since $x \succ_S z \succ_S w$, $x \succ_S w$, and only pair $\{x, w\}$ is labeled 1 in $\{x, z, w\}$ (because $\{x, z\}$ is not labeled 1 by the assumption of case (b) and $\{z, w\}$ belongs to C_w in which $\{y, z\}$ is labeled 1), this falls into case (a) for the triple $\{w, x, z\}$.

Thus, case (b) is also impossible. Hence, $\succ_1$ has no 3-cycle, and is therefore transitive. □

Remark 4. The existence of an oriented labeling is not necessary for 3-party representability. Consider the social preference relation $\succ_S$ illustrated in Figure 5. It can be checked that no labeling of $\succ_S$ is oriented. Figure 5 illustrates a non-oriented labeling. Yet, this labeling defines a 3-party system representing $\succ_S$. □

Remark 5. A *minimum reversing set* of a social preference relation $\succ_S$ is a subset of policy pairs of minimum size such that reversing all pairs in the set makes $\succ_S$ transitive. Proposition 2 implies that if $\succ_S$ admits an oriented labeling L, then the three sets L_i, $i = 1, 2, 3$, are minimum reversing sets of $\succ_S$. Each party's preference relation obtained through an oriented labeling is therefore closest to the social preference relation among all transitive preferences, if we measure the distance between two preference relations by the number of policy pairs on which they disagree.[5] □

[5] Barthélemy, Hudry, Isaak, Roberts, and Tesman (1995) shows that every policy pair in a minimum reversing set of $\succ_S$ belongs to some 3-cycle of $\succ_S$.

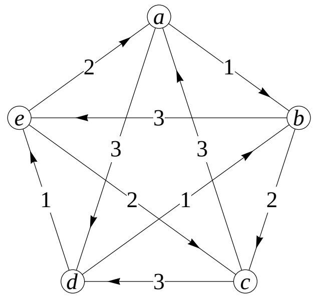

Figure 5: Social preference having no oriented labeling

Remark 6. Proposition 2 shows that an oriented labeling, if exists, provides a method to construct a set of three voters that induces a given preference. Erdös and Moser (1964) present a different construction that needs more voters, but always succeeds. □

5 A sufficient condition for 3-party representability

We have seen that the existence of an oriented labeling is sufficient for 3-party representability. In this section we provide a sufficient condition for a social preference relation to have an oriented labeling, which restricts more explicitly the structure of the social preference relation. The condition says that the set of 3-cycles of the social preference relation contains no "closed path"; i.e., it has a "forest"-like structure. This structure of 3-cycles enables us to sequentially implement an oriented labeling.

A *closed path* of 3-cycles of a social preference relation $\succ_S$ is a cyclic sequence of distinct 3-cycles of $\succ_S$, $C = (C_1, C_2, ..., C_k, C_1)$ with $k \geq 2$, such that:

(C1) for each j, C_j and C_{j+1} share a policy pair; and

(C2) if C_j and C_{j+1} share a policy pair $\{x_j, y_j\}$ for all j, then $\{x_j, y_j\} \neq \{x_h, y_h\}$ for $j \neq h$.[6]

The social preference relation illustrated in Figure 5 contains the closed path of 3-cycles $((a, b, c, a), (b, c, d, b), (c, d, e, c), (d, e, a, d), (e, a, b, e), (a, b, c, a))$. In

[6]Here all subscripts are mod k.

contrast, in Figure 1, the sequence $((a, b, c, a), (a, b, d, a), (a, b, c, a))$ is not a closed path of 3-cycles: while it satisfies (C1), it does not satisfy (C2).

Proposition 3. *If a social preference relation $\succ_S$ has no closed path of 3-cycles, then $\succ_S$ admits an oriented labeling, and hence $\succ_S$ is 3-party representable.*

Proof. Recall the graph $G = G(\succ_S)$ defined in Remark 3. A labeling in our sense naturally translates into a labeling (or a "3-coloring") over the *verteces* of G, i.e., those policy pairs that belong to 3-cycles of $\succ_S$.

We list properties of the graph G:

(G1) Every 3-cycle (x, y, z, x) of $\succ_S$ corresponds exactly to the (non-directed) 3-cycle $(\{x, y\}, \{y, z\}, \{z, x\}, \{x, y\})$ of G.

(G2) Every vertex and every edge of G belong to some 3-cycle of G.

(G3) Any two distinct 3-cycles of G have at most one vertex in common; and this occurs if and only if the corresponding 3-cycles of $\succ_S$ share a policy pair.

Properties (G1) and (G2) are obvious from the definition of G. (G3) holds since two distinct 3-cycles of $\succ_S$ can share at most two policies (i.e., at most one policy pair).

These properties imply that the graph G typically has the form as illustrated in Figure 6.

Suppose $\succ_S$ has no closed path of 3-cycles.

Then G can contain only 3-cycles. To see this, suppose G has a cycle C^G of length more than 3. As Figure 6 indicates, the cycle C^G must run through several 3-cycles of G, and these 3-cycles constitute a cyclic sequence. This cyclic sequence of 3-cycles represents a closed path of 3-cycles of $\succ_S$, a contradiction.

Thus G has the form as depicted in Figure 7: G comprises possibly multiple "trees" each of which is made up by connected 3-cycles. For each such tree, we can sequentially implement an oriented labeling as follows:

- Step 1. Choose any 3-cycle C_1^G in G and label the verteces of C_1^G (of course, subject to (L1) and (L2)).
- Step $t \geq 2$. For any 3-cycle C_t^G of G which has been partially labeled at Step $t-1$, label the remaining verteces of C_t^G in such a way that (O1) holds for the labels assigned so far.
- Continue until there is no 3-cycle that has exactly one labeled vertex.

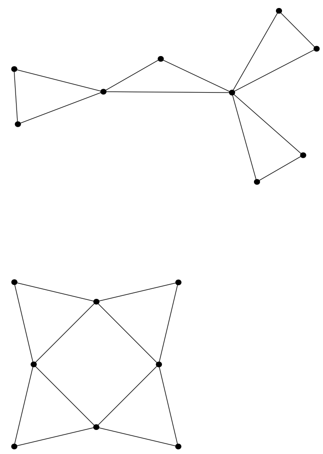

Figure 6: Graph G

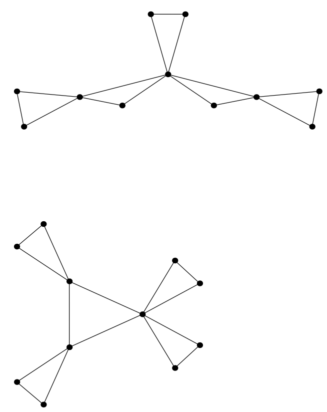

Figure 7: Graph G without cycles of lengths more than 3

It is clear from Figure 7 that every 3-cycle of G for which we must complete labeling at Step t has exactly one vertex that has been labeled in Step $t-1$. As a result, since the steps proceed as if spreading out along "branches" of the tree from the initial ("root") 3-cycle, at each step $t \geq 2$ we face 3-cycles each having two unlabeled verteces with full degree of freedom to label. It is therefore possible to complete each step subject to (O1).

A round of sequential labeling described above establishes an oriented labeling of a connected component of G, and we can repeat such a round for all components. The resulting labeling of G expresses an oriented labeling of $\succ_S$. □

Remark 7. There must be a threshold $\bar{n}$ of the number n of policies below which all social preference relations are 3-party representable. We do not know the exact value of $\bar{n}$. By cardinality comparison, any social preference relation over n policies can be induced by some 3-party system only if $2^{n(n-1)/2} \leq (n!)^3$.[7] The maximum n satisfying this inequality is 18. Thus $\bar{n} \leq 18$.

There is an explicit (but somewhat artificial) example of social preference relation over 19 policies that is not 3-party representable. The graph-expression of this preference is the "Paley tournament" on 19 verteces, denoted T_{19}. Recall the property of 3-voter majority preferences proved by Alon et al. (2006) which we

[7] I owe this idea to Stearns (1959).

mentioned in Introduction. It can be checked (as Graham and Spencer (1971) claim) that T_{19} does not satisfy Alon et al's condition. □

6 Conclusion

In this paper we studied the range of social preferences that a three-party legislature can realize through majority voting. We presented a procedure to construct a three-party system that realizes a given social preference relation. We applied the construction procedure to establish a sufficient condition for a social preference relation to be compatible with some three-party system. The sufficient condition says that the social preference relation has no closed path of 3-cycles. This condition reveals a region of social preferences that a three-party legislature can realize, but a legislature with a single-party majority cannot realize.

We have imposed no restriction on the composition of a legislature, apart from the bound on the number of parties. But in reality a society is endowed with a set of established parties. The preferences and seat shares of these parties may be limited to certain ranges, irrespective of institutional arrangements the society can make. Under such constraints, comparison of party systems may not be so simple as in this paper. In particular, it is possible that within the feasible set of party profiles, some party system with a single-party majority generates a preference relation that is closer to the social preference relation than any multi-party system generates. In such situations which type of party system has advantage in representing the society depends on the social preferences. Future research may extend the framework of this paper to include such cases.

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