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Inference for the jump part of quadratic variation of Itô semimartingales

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Abstract

Recent research has focused on modelling asset prices by Itô semimartingales. In such a modelling framework, the quadratic variation consists of a continuous and a jump component. This paper is about inference on the jump part of the quadratic variation, which can be estimated by the difference of realised variance and realised multipower variation. The main contribution of this paper is twofold. First, it provides a bivariate asymptotic limit theory for realised variance and realised multipower variation in the presence of jumps. Second, this paper presents new, consistent estimators for the jump part of the asymptotic variance of the estimation bias. Eventually, this leads to a feasible asymptotic theory which is applicable in practice. Finally, Monte Carlo studies reveal a good finite sample performance of the proposed feasible limit theory.

Keywords: Quadratic variation, Itô semimartingale, stochastic volatility, jumps, realised variance, realised multipower variation, high–frequency data.
JEL classification: C13, C14, G10, G12.

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1 Introduction

Estimation and inference of the variation of asset prices is one of the key tasks in financial econometrics. In this paper, we focus on very general models for logarithmic asset prices, namely on Itô semimartingales. Such processes are special cases of real–valued semimartingales in the sense that they can be written as $Y = (Y_t)_{t\geq 0}$ with

$$Y_t = Y_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + L_t, \qquad (1)$$

on a probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Here, $(W_t)_{t\geq 0}$ is a one–dimensional Brownian motion, $(b_t)_{t\geq 0}$ is a locally bounded predictable process, $(\sigma_t)_{t\geq 0}$ is a càdlàg adapted process and $(L_t)_{t\geq 0}$ is a fairly general jump process and will be specified more precisely below.

The variation of asset prices is often measured by means of the quadratic variation which is given by

$$[Y]_t = [Y]_t^c + [Y]_t^d, \qquad \text{where} \qquad [Y]_t^c = \int_0^t \sigma_s^2 ds, \qquad \text{and} \qquad [Y]_t^d = \sum_{0\leq s\leq t} (\Delta L_s)^2,$$

denote the continuous and discontinuous (or jump) parts of the quadratic variation, respectively.

While inference on the integrated variance has been studied extensively in the last decade, see e.g. Barndorff-Nielsen & Shephard (2002), inference on the quadratic variation in the presence of jumps and inference on the jump part of the quadratic variation, in particular, has not been studied yet. So far, it has only been possible to gain information on the jump part of the quadratic variation indirectly, i.e. by means of tests for jumps as studied by e.g. Barndorff-Nielsen & Shephard (2006), Aït-Sahalia & Jacod (2006) and Jacod & Todorov (2007). However, this paper introduces an explicit, non–parametric method, based on so–called *in–fill asymptotics*, for making inference on the jump part of the quadratic variation.

Our first steps will follow the methodology of Barndorff-Nielsen & Shephard (2006), who exploited the fact that jumps in the asset price are reflected in a jump part of the quadratic variation and vice versa. So their main idea was to compare two measures of variance: one which is not robust to jumps, a quantity called *realised variance* (see e.g. Comte & Renault (1998), Barndorff-Nielsen & Shephard (2002), Andersen, Bollerslev, Diebold & Labys (2001), Jacod (2007*a*)), that estimates the entire quadratic variation of the price process, and one which is robust to jumps, called *realised multipower variation* (see e.g. Barndorff-Nielsen & Shephard (2004), Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006), Barndorff-Nielsen, Shephard & Winkel (2006), Woerner (2006)). Jacod (2006)), and only estimates the continuous part of the quadratic variation. By using the difference of these two quantities, one obtains a consistent estimator for the jump part of the quadratic variation. In order to make inference on the jump part of quadratic variation, one has to derive the asymptotic distribution of the difference of realised variance and realised multipower variation or, more generally, their joint asymptotic distribution in the presence of jumps. This is exactly the task we tackle in this paper.

The remaining part of the paper is structured as follows. Section 2 introduces the notation and the main model assumptions. Section 3 reviews the most important facts about realised variance and realised multipower variation. Section 4 contains the main contribution of this paper. First, we sketch some of the important theoretical work by Jacod (2006, 2007*a*) on univariate asymptotic results for realised variance and realised multipower variation. Then, the main result is presented: the asymptotic distribution of a bivariate process of realised variance and realised multipower variation in the presence of jumps. From this result we deduce the asymptotic distributions of the difference, the ratio and the logarithmic difference of realised variance and realised multipower variation. At first, all these limit theorem are *infeasible* in the sense that the asymptotic variances of the estimation bias are not observable. So in a next step, we replace them by consistent estimators, which eventually leads to *feasible* limit theorems. In order to assess the finite sample performance of the feasible asymptotic theory, we carry out a detailed Monte Carlo study in Section 5. Finally, Section 6 concludes the paper and gives some prospect on future research. The proofs of the theorem and the tables with the results from the simulation study are given in the Appendices.

2 Setup

This section sets up the notation and introduces the model assumptions, which are essentially taken from Jacod (2007*a*). The logarithmic asset price is supposed to be a real–valued *Itô semimartingale* $Y = (Y_t)_{t\geq 0}$, which is defined on a probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Recall that any semimartingale has predictable characteristic (B, C, ν), say, where ν is the compensator of the jump measure μ of Y, $C = \langle Y^c \rangle$, where Y^c is the continuous martingale part of Y, and the drift B depends on the choice of a truncation function κ, say, which is supposed to be continuous, bounded with compact support and $\kappa(x) = x$ on a neighbourhood of 0. Further let $\kappa'(x) = x - \kappa(x)$.

An *Itô semimartingale* is defined as a semimartingale whose characteristics are absolutely continuous with respect to the Lebesgue measure, i.e.

$$B_t = \int_0^t b_u du, \qquad C_t = \int_0^t c_u du, \qquad \nu(dt, dx) = dt F_t(dx). \tag{2}$$

Clearly, for $\sigma_t = \sqrt{c_t}$, every Itô semimartingale can be written as

$$Y_t = Y_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \kappa(\delta) \star (\underline{\mu} - \underline{\nu})_t + \kappa'(\delta) \star \underline{\mu}_t, \tag{3}$$

where $\delta : \Omega \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is predictable and such that $F_t(\omega, dx)$ is the image of the Lebesgue measure on $\mathbb{R}$ by the map $x \mapsto \delta(\omega, t, x)$, and W is a Brownian motion and $\underline{\mu}$ a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ on the filtered space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and the predictable compensator of $\underline{\mu}$ is $\underline{\nu}(ds, dx) = ds \otimes dx$, where it might be necessary to enlarge the original space to accommodate $(W, \underline{\mu})$. In the following we will often use a function ϕ_s, which is defined by

$$\phi_s(x) = \begin{cases} \min(1, |x|^s), & \text{if } 0 < s < \infty, \\ \mathbb{I}_{\mathbb{R}\setminus\{0\}} & \text{if } s = 0. \end{cases}$$

Now we can introduce the model assumptions (as given in Jacod (2007*a*)).

Hypothesis (H) Y is an Itô semimartingale and in (2) the processes $(b_t)_{t\geq 0}$ and $(F_t(\phi_2))_{t\geq 0}$ are locally bounded predictable (where $F_t(f) = \int f(x) F_t(dx)$), and the process $(c_t)_{t\geq 0}$ is càdlàg adapted.

At first sight, this assumption might look quite technical. However, it is a fairly natural extension of standard asset price models: A Brownian semimartingale $\int_0^t \sigma_s dW_s$ is one of the main building blocks for asset price models, since it allows for stochastic volatility and can incorporate the leverage effect, when σ and W are not assumed to be independent. In order to allow for jumps, recent research has then focused on adding a jump component, e.g. a compound Poisson process or a more general pure jump Lévy process to a Brownian semimartingale. An Itô semimartingale is a further extension of such a model in the sense that it drops the possibly unrealistic assumption of a Lévy process, which has independent and stationary increments.

Another assumption is concerned with the jump part of the semimartingale.

Hypothesis (K) (H) holds and the coefficient δ (see (3)) satisfies $|\delta(\omega, t, x)| \leq \gamma_k(x)$ for all $t \leq T_k(\omega)$, where γ_k denote some deterministic functions on $\mathbb{R}$ which satisfy $\int \phi_2 \circ \gamma_k(x) dx < \infty$, and (T_k) are stopping times increasing to $+\infty$.

Furthermore, we need an assumption on the volatility process and on the activity of the jumps of Y.

Hypothesis (L-s) (for $s \in [0, 2]$) Assume that the probability space supports a second Brownian motion $W' = (W'_t)_{t\geq 0}$, which is independent of W. Further, (H) holds and the volatility process σ has the form

$$\sigma_t = \sigma_0 + \int_0^t \widetilde{b}_u du + \int_0^t \widetilde{\sigma}_u dW_u + \int_0^t \widetilde{\sigma}'_u dW'_u + \kappa(\widetilde{\delta}) \star (\underline{\mu} - \underline{\nu})_t + \kappa'(\widetilde{\delta}) \star \underline{\mu}_t,$$

and

- the process $(\widetilde{b}_t)$ is optional and locally bounded;
- the processes (b_t), $(\widetilde{\sigma}_t)$, $(\widetilde{\sigma}'_t)$ are adapted left–continuous with right limits in t, and locally bounded;

- the functions $\delta(\omega, t, x)$ and $\tilde{\delta}(\omega, t, x)$ are predictable and left–continuous with right limits in t. Also, $|\delta(\omega, t, x)| \leq \gamma_k(x)$ and $|\tilde{\delta}(\omega, t, x)| \leq \tilde{\gamma}_k(x)$ for all $t \leq T_k(\omega)$, where $\gamma_k, \tilde{\gamma}_k$ are deterministic functions on $\mathbb{R}$ with $\int \phi_s \circ \gamma_k(x)dx < \infty$ (where we define $0^0 = 0$) — note that this is the condition where the s comes in — and $\int \phi_2 \circ \tilde{\gamma}_k(x)dx < \infty$. Furthermore, (T_k) denotes a sequence of stopping times increasing to $+\infty$.

So, under the assumption (L-s), s can be regarded as a generalised Blumenthal–Getoor index of Y, which measures the activity of the jumps of the Itô semimartingale Y, and the volatility process is given by a semimartingale. Note that we assume in (L-s) that $s \in [0, 2]$. If $s \leq s' \leq 2$, then (L-s)⇒ (L-s')⇒ (K) ⇒(H). Also note that (L-0) implies that X has locally finitely many jumps and if X is continuous, then all hypotheses (L-s) are identical for all $s \in [0, 2]$ (see Jacod (2007*a*, p.6)). Finally, we formulate a hypothesis which guarantees that the Brownian semimartingale component is nowhere degenerate.

Hypothesis (H') Hypothesis (H) holds and (σ_t^2) and (σ_{t-}^2) do not vanish.

For our asymptotic theory, we need some further notation, which follows Jacod (2007*a*)'s framework. Let $(\Omega', \mathcal{A}', \mathbb{P}')$ denote an auxiliary space which supports two Brownian motions $\overline{W}$ and $\widetilde{W}$, two sequences of $\mathcal{N}(0, 1)$ random variables, denoted by (U_p) and (U'_p) and, further, a sequence of random variables (ξ_p) which are uniformly distributed on $[0, 1]$. All these processes are assumed to be mutually independent. Now we extend our original probability space and we write

$$\widetilde{\Omega} = \Omega \times \Omega', \qquad \widetilde{\mathcal{A}} = \mathcal{A} \otimes \mathcal{A}', \qquad \widetilde{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}'.$$

One can now extend, in the obvious way, the variables $Y_t,\ b_t, \ldots$ defined on Ω and $\overline{W},\ \widetilde{W},\ U_p, \ldots$ defined on Ω' to the product space (without change of notation). Let $\widetilde{\mathbb{E}}$ denote the expectation with respect to $\widetilde{\mathbb{P}}$. Further, let (T_p) denote stopping times which are an enumeration of the jump times of Y. Finally, we write $(\widetilde{\mathcal{F}}_t)$ for the smallest right–continuous filtration of $\widetilde{\mathcal{A}}$ which contains $(\mathcal{F}_t)$ and with respect to which $\overline{W}$ is adapted and, further, such that U_p, U'_p and ξ_p are $\widetilde{\mathcal{F}}_{T_p}$–measurable for all p.

Straightforwardly, $\overline{W}$ and $\widetilde{W}$ are $(\widetilde{\mathcal{F}}_t)_{t\geq 0}$–Brownian motions under $\widetilde{\mathbb{P}}$, which also holds for W and W'. Further $\underline{\mu}$ is a Poisson measure with compensator $\underline{\nu}$ for the bigger filtration.

3 Review of realised variance and realised multipower variation

After having introduced the admittedly quite tedious notation for the continuous–time price process, we now turn our attention to its discrete–time observations. Let us assume that we observe the process Y over an interval $[0,t]$ at times $i\Delta_n$ for $\Delta_n > 0$ and $i = 0, \ldots, [t/\Delta_n]$. We will always assume that $\Delta_n \to 0$ as $n \to \infty$. So for the discretely observed increments, we write

$$\Delta_i^n Y = Y_{i\Delta_n} - Y_{(i-1)\Delta_n}, \quad \text{for } i = 1, \ldots [t/\Delta_n].$$

In practice, these increments are used to construct estimators for the variance or integrated variance. For example, it is well–known that the *realised variance*, which is the sum of the squared increments, given by

$$RV_t^n = \sum_{i=1}^{[t/\Delta_n]} (\Delta_i^n Y)^2,$$

estimates the quadratic variation of the underlying process consistently, i.e. $RV_t^n \xrightarrow{ucp} [Y]_t$, as $n \to \infty$, where the convergence is uniformly on compacts in probability (ucp) (see Protter (2004, p. 57), Andersen, Bollerslev, Diebold & Ebens (2001) and Barndorff-Nielsen & Shephard (2002)).

Besides, one can use the *realised bipower variation* (as defined by Barndorff-Nielsen & Shephard (2004, 2006)) for estimating the continuous part of the quadratic variation of Itô semimartingales (see Jacod (2006)). Let $\mu_r = \mathbb{E}|U|^r$, for $U \sim N(0,1)$. Then, one obtains

$$\mu_1^{-2} \sum_{i=1}^{[t/\Delta_n]-1} |\Delta_i^n Y| \left|\Delta_{i+1}^n Y\right| \xrightarrow{ucp} [Y^c]_t = \int_0^t \sigma_s^2 ds, \quad \text{as } n \to \infty.$$

This concept can be further generalised to realised multipower variation (see e.g. Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006) for a treatment of realised multipower variation in the absence of jumps and Woerner (2006) and Jacod (2006) for the corresponding results in the presence of jumps). Let $\mathbf{r} = (r_1, \ldots, r_I)$ be a multi–index with $r_i > 0$. Further, we write $|\mathbf{r}| = r_1 + \cdots + r_I$ and $\mathbf{r}_+ = \max_{1\leq i\leq I} r_i$ and $\mathbf{r}_- = \min_{1\leq i \leq I} r_i$. Let $\mathbf{r}_+ < 2$, then

$$\Delta_n^{1-|\mathbf{r}|/2} \mu_{\mathbf{r}}^{-1} \sum_{i=1}^{[t/\Delta_n]-I+1} \prod_{j=1}^{I} |\Delta_{i+j-1}^n Y|^{r_j} \xrightarrow{ucp} \int_0^t |\sigma_u|^{|\mathbf{r}|} du, \quad \text{as } n \to \infty,$$

where $\mu_{\mathbf{r}} = \prod_{j=1}^I \mu_{r_j}$. Now we define

$$RMPV(\mathbf{r})_t^n = \frac{[t/\Delta_n]}{[t/\Delta_n] - I + 1} \Delta_n^{1-|\mathbf{r}|/2} \mu_{\mathbf{r}}^{-1} \sum_{i=1}^{[t/\Delta_n]-I+1} \prod_{j=1}^{I} |\Delta_{i+j-1}^n Y|^{r_j}. \tag{4}$$

Since we want to study the difference of $RMPV(\mathbf{r})_t^n$ and RV_t^n in the following section, we include the factor $[t/\Delta_n]/([t/\Delta_n] - I + 1)$ in the definition of realised multipower variation above. This accounts for the fact that there are only $([t/\Delta_n] - I + 1)$ terms in the sum on the right hand side of (4), whereas there are $[t/\Delta_n]$ summands in the realised variance case. Making such an adjustment avoids the problem of introducing a finite sample bias by comparing to similar sums with a different number of summands each. Clearly, if $\mathbf{r}_+ < 2$, then

$$RMPV(\mathbf{r})_t^n \overset{ucp}{\to} \int_0^t |\sigma_u|^{|\mathbf{r}|} du, \quad \text{as } n \to \infty.$$

Note that if $|\mathbf{r}| = 2$, then we simply get $\Delta_n^{1-|\mathbf{r}|/2} = 1$. In particular, we are interested in realised multipower variations with equal power r_i. So we define for $k, I \in \mathbb{N}$:

$$RMPV(k;I)_t^n = \frac{[t/\Delta_n]}{[t/\Delta_n] - I + 1} \Delta_n^{1-k/2} \mu_{k/I}^{-I} \sum_{i=1}^{[t/\Delta_n]-I+1} \prod_{j=1}^{I} |\Delta_{i+j-1}^n Y|^{k/I}.$$

Then, for $k/I < 2$,

$$RMPV(k;I)_t^n \overset{ucp}{\longrightarrow} \int_0^t |\sigma_u|^k du, \quad \text{as } n \to \infty.$$

For $k = 2$, we have $RMPV(2;I)_t^n \overset{ucp}{\longrightarrow} \int_0^t \sigma_u^2 du$, as $n \to \infty$, and, hence,

$$RV_t^n - RMPV(2;I)_t^n \overset{ucp}{\longrightarrow} [Y]_t^d, \quad \text{as } n \to \infty.$$

So, the difference of realised variance and realised multipower variation is a consistent estimator for the jump part of the quadratic variation. In order to make inference on $[Y]^d$, we derive an asymptotic theory for the difference of realised variance and realised multipower variation. More generally, we study the asymptotic properties of the bivariate vector of realised variance and realised multipower variation.

4 Central limit theorems in the presence of jumps

Let Y be the real–valued Itô semimartingale as defined above. We are interested in studying the asymptotic properties of the centered bivariate vector

$$\frac{1}{\sqrt{\Delta_n}} \begin{pmatrix} RV_t^n - [Y]_t \\ RMPV(2;I)_t^n - [Y]_t^c \end{pmatrix}. \tag{5}$$

Remark In the limit results below, we will use the concept of stable convergence in law, which is a stronger convergence than convergence in law. It can be defined in the following way (see e.g. Jacod & Shiryaev (2003) and Barndorff-Nielsen, Graversen, Jacod & Shephard (2006)). Let X_n denote a sequence of random variables on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $(U, \mathcal{U})$ denote a Polish space. If there is a probability measure μ defined on the extended space $(\Omega \times U, \mathcal{A} \times \mathcal{U})$ such that for every random variable Z on $(\Omega, \mathcal{A}, \mathbb{P})$ and for every bounded and continuous function g on U we have

$$\mathbb{E}\left(Zg(X_n)\right) \to \int Z(\omega)g(x)\mu(d\omega, dx), \text{ as } n \to \infty,$$

then we say that X_n converges *stably in law*.

4.1 Univariate central limit theorems

Jacod (2006, 2007*a*) has proven the univariate limit results for both components. So, for the realised variance, let us assume that (L-2) is satisfied. From Jacod (2007*a*, Theorem 2.11 (ii)), we get

$$\frac{1}{\sqrt{\Delta_n}}\left(RV_t^n - [Y]_{\Delta_n[t/\Delta_n]}\right) \overset{stably\ in\ law}{\longrightarrow} \Lambda_t^{(1)} + \Lambda_t^{(2)}, \text{as } n \to \infty, \tag{6}$$

where the convergence is stably in law as a process. The limiting process is given by $\Lambda_t^{(1)} + \Lambda_t^{(2)}$, where

$$\Lambda_t^{(1)} = \sqrt{2}\int_0^t \sigma_u^2 d\overline{W}_u, \qquad \Lambda_t^{(2)} = 2\sum_{p:\ T_p \le t} \Delta Y_{T_p}\left(\sqrt{\xi_p}U_p\sigma_{T_p-} + \sqrt{1-\xi_p}\,U_p'\sigma_{T_p}\right), \tag{7}$$

with $\overline{W}$, T_p, ξ, U, U' as defined at the end of Section 2. Furthermore, we know from Jacod (2007*a*) that stable convergence in law as a process only holds when the discretised process $[Y]_{\Delta_n[t/\Delta_n]}$ is used in (6). However, $\frac{1}{\sqrt{\Delta_n}}\left(RV_t^n - [Y]_t\right)$ converges *finite–dimensionally stably in law* (abbreviated by *fin. stably in law*) to the limit described above (see Jacod (2007*a*, Remark 2.14)). But the latter result will be sufficient for us since we are interested in making inference on the jump part of the quadratic variation at a fixed time t. Conditionally on $\mathcal{A}$, $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are *independent* and $\Lambda^{(1)}$ is a *martingale* with *Gaussian law*, and if Y and σ do not jump together, $\Lambda^{(2)}$ is also a martingale with Gaussian law. Their variances are given by ((Jacod 2007*a*, p. 8))

$$\widetilde{\mathbb{E}}\left(\left(\Lambda_t^{(1)}\right)^2\middle|\mathcal{A}\right) = 2\int_0^t \sigma_u^4 du, \quad \widetilde{\mathbb{E}}\left(\left(\Lambda_t^{(2)}\right)^2\middle|\mathcal{A}\right) = 2\sum_{p:\ T_p\le t}\left(\Delta Y_{T_p}\right)^2\left(\sigma_{T_p}^2 + \sigma_{T_p-}^2\right).$$

So, conditionally on $\mathcal{A}$, the asymptotic variance of the bias between realised variance and quadratic variation is given by

$$2\int_0^t \sigma_u^4 du + 2\sum_{p:\ T_P\le t}\left(\Delta Y_{T_p}\right)^2\left(\sigma_{T_p}^2 + \sigma_{T_p-}^2\right). \tag{8}$$

Finally note that in the absence of jumps, the limit is given by $\Lambda_t^{(1)}$, which is a well–known result, e.g. Jacod (1994), Jacod & Protter (1998) and Barndorff-Nielsen & Shephard (2002). In the following, we will call a stochastic process on the extended probability space $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mathbb{P}})$

mixed normal (abbreviated by the capital letters MN), if its distribution is, conditional on $\mathcal{A}$, Gaussian. Hence, in the absence of common jumps of σ and Y, $\Lambda_t^{(1)} + \Lambda_t^{(2)}$ is mixed normal.

Now we turn our attention to realised multipower variation, whose asymptotic distribution in the presence of jumps has first been derived by Woerner (2006). A later study by Jacod (2006, Theorem 6.2) contains the following result. Assume that (L-s) holds for some $s < 1$ and that we have (H'). Furthermore let $\mathbf{r}$ be a multi–index such that $\frac{s}{2-s} < \mathbf{r}_- \leq \mathbf{r}_+ < 1$. Then, as $n \to \infty$,

$$\frac{1}{\sqrt{\Delta_n}} \left(RMPV(\mathbf{r})_t^n - \int_0^t |\sigma_u|^{|\mathbf{r}|} du \right) \overset{stably\ in\ law}{\longrightarrow} \mu_{\mathbf{r}}^{-1} \sqrt{A(\mathbf{r})} \int_0^t |\sigma_u|^{|\mathbf{r}|} d\widetilde{W}_u, \tag{9}$$

stably in law as a process, for a new, independent Brownian motion $\widetilde{W}$, where

$$A(\mathbf{r}) = \prod_{i=1}^{I} \mu_{2r_i} - (2I-1) \prod_{i=1}^{I} \mu_{r_i}^2 + 2 \sum_{i=1}^{I-1} \prod_{j=1}^{i} \mu_{r_j} \prod_{j=I-i+1}^{I} \mu_{r_j} \prod_{j=1}^{I-i} \mu_{r_j + r_{j+i}}.$$

Remark The central limit result (9) does not hold for bipower variation of power 1, i.e. for $RMPV(2,2)$. We suppose that it is possible to derive a central limit theorem for this realised bipower variation in the presence of jumps. However, the central limit theorem for realised bipower variation will differ from the ones for realised tripower, realised quadpower etc., in the sense that the limit process will exhibit a jump component in addition to the Brownian semimartingale, as mentioned in Barndorff-Nielsen, Shephard & Winkel (2006, Section 3.1). So we expect to obtain a central limit result which is similar to the one for realised variance, however, the jump part of the limit process for realised bipower variation will probably not be Gaussian anymore. This aspect will be studied in more detail in future research.

In the next section, we combine these two results and derive a bivariate limit result, which is the main contribution of this paper.

4.2 Main results

Let $(Y_t)_{t \geq 0}$ denote a one–dimensional Itô semimartingale.

Theorem 4.1 *Assume (L-s) for some $s < 1$, (H') and let $\mathbf{r}$ be a multi–index such $\frac{s}{2-s} < \mathbf{r}- \leq \mathbf{r}+ < 1$. Then, as $n \to \infty$,*

$$\frac{1}{\sqrt{\Delta_n}} \begin{pmatrix} RV_t^n - [Y]_{\Delta_n[t/\Delta_n]} \\ RMPV(\mathbf{r})_t^n - \int_0^t |\sigma_u|^{|\mathbf{r}|} du \end{pmatrix}$$
$$\overset{stably\ in\ law}{\longrightarrow} \begin{pmatrix} \sqrt{2} \int_0^t \sigma_u^2 d\overline{W}_u + 2 \sum_{p:\, T_p \leq t} \Delta Y_{T_p} \left(\sqrt{\xi_p} U_p \sigma_{T_p-} + \sqrt{1-\xi_p}\, U_p' \sigma_{T_p} \right) \\ \sqrt{2} \int_0^t |\sigma_u|^{|\mathbf{r}|} d\overline{W}_u + \sqrt{\theta_{\mathbf{r}}} \int_0^t |\sigma_u|^{|\mathbf{r}|} d\widetilde{W}_u \end{pmatrix},$$

where the convergence is stable in law as a process and $\theta_{\mathbf{r}} = (\mu_{\mathbf{r}}^{-1} \sqrt{A(\mathbf{r})})^2 - 2$.

Proof The proof is given in the Appendix. □

If σ and Y do not jump together, the limit process is mixed normal. Note that in that case $\sigma_{T_{p-}} = \sigma_{T_p}$ since T_p are the jump times of Y.

Remark The one–dimensional limit result for the multipower variation holds as soon as (L-s) for some $s < 1$, (H') holds and $\frac{s}{2-s} < \mathbf{r}- \leq \mathbf{r}+ < 1$. In order to obtain the limit result for the realised variance, we need the assumption (L-2) which is clearly implied by (L-s) for some $s < 1$.

Corollary 4.2 *Assume (L-s) for some $s < 1$, (H') and that Y and σ have no common jumps. For $I \in \mathbb{N}$ with $2 < I < \frac{2}{s}(2-s)$, we obtain, as $n \to \infty$:*

$$\frac{1}{\sqrt{\Delta_n}} \begin{pmatrix} RV_t^n - [Y]_t \\ RMVP(2;I)_t^n - \int_0^t \sigma_u^2 du \end{pmatrix} \xrightarrow{fin.\ stably\ in\ law} \begin{pmatrix} \sqrt{2}\int_0^t \sigma_u^2 d\overline{W}_u + \sqrt{2}\ \sum_{p:\ T_p \leq t} \Delta Y_{T_p} \sigma_{T_p} \left(\sqrt{\xi_p} U_p + \sqrt{1-\xi_p}\ U_p'\right) \\ \sqrt{2}\int_0^t \sigma_u^2 d\overline{W}_u + \sqrt{\theta_I}\int_0^t \sigma_u^2 d\widetilde{W}_u \end{pmatrix}, \quad (10)$$

where (10) has, conditionally on $\mathcal{A}$, Gaussian law with zero mean and variance

$$\Sigma_t^{(I)} = \begin{pmatrix} 2\int_0^t \sigma_u^4 du + 4\ \sum_{p:\ T_p \leq t} \left(\Delta Y_{T_p}\right)^2 \sigma_{T_p}^2 & 2\int_0^t \sigma_u^4 du \\ 2\int_0^t \sigma_u^4 du & (2+\theta_I)\int_0^t \sigma_u^4 du \end{pmatrix},$$

where $\theta_I = \mu_{2/I}^{-2I}\omega_I^2 - 2$ and $\omega_I^2 = \mu_{4/I}^I + (1-2I)\mu_{2/I}^{2I} + 2\sum_{j=1}^{I-1}\mu_{4/I}^{I-j}\ \mu_{2/I}^{2j}$.

Proof This result is a direct consequence of Theorem 4.1 when we set $r_j = 2/I$ for $j = 1, \ldots, I$. □

Note that we assume (L-s) for $s < 1$ and and integer I with $2 < I < \frac{2}{s}(2-s)$. Clearly, if $s = 0$, the assumptions are satisfied for all integers $I \geq 3$. In general, higher powers can only be used if the generalised Blumenthal–Getoor index is fairly close to 0 as shown in Table 1. Hence, our assumption on the Blumenthal–Getoor index might look quite restrictive at first

I	3	4	5	10	20	I
$s <$	4/5	2/3	4/7	1/3	2/11	$4/(I+2)$

Table 1: Relationship between s and I.

sight. However, very recent work by Cont & Mancini (2007) indicates that the assumption that the Blumenthal–Getoor index is smaller than 1, which implies jumps of possibly infinite activity, but finite variation, seems to be in line with recent empirical findings on high frequency asset price data.

In the remaining part of this section, we derive some central limit results for the difference, the ratio and the logarithmic difference of realised variance and realised multipower variation. These results follow directly from From Theorem 4.1 and the bivariate delta method for stable convergence (see Dette et al. (2006)). In order to make inference on the jump part of the quadratic variation, we have to study the asymptotic distribution of the difference of realised variance and realised multipower variation. This result is important in applications and can, hence, be regarded as one of the key results of this paper.

Corollary 4.3 *Assume (L-s) for some $s < 1$, (H') and that Y and σ have no common jumps. For $I \in \mathbb{N}$ with $2 < I < \frac{2}{s}(2-s)$, we obtain, as $n \to \infty$:*

$$\frac{1}{\sqrt{\Delta_n}}(RV_t^n - RMPV(2;I)_t^n - [Y]_t^d) \overset{fin.\ stably\ in\ law}{\longrightarrow} S_t^{(1)}, \tag{11}$$

where $S_t^{(1)}$ has, conditionally on $\mathcal{A}$, Gaussian law with zero mean and variance given by

$$\theta_I \int_0^t \sigma_u^4 du + 4 \sum_{p:\ T_P \le t} \left(\Delta Y_{T_p}\right)^2 \sigma_{T_p}^2. \tag{12}$$

Remark If σ and Y have common jumps, the limit process of the left hand side of (11) can still be derived from Theorem 4.1. However, its distribution is not Gaussian, which makes it slightly more difficult to use in practice. Nevertheless its distribution can be simulated.

Furthermore, we might be interested in making inference on the ratio of the jump part of the quadratic variation and the entire quadratic variation. The corresponding asymptotic theory is given in the following Corollary.

Corollary 4.4 *Assume (L-s) for some $s < 1$, (H') and that Y and σ have no common jumps. For $I \in \mathbb{N}$ with $2 < I < \frac{2}{s}(2-s)$, we obtain, as $n \to \infty$:*

$$\frac{1}{\sqrt{\Delta_n}}\left(\frac{RV_t^n - RMPV(2;I)_t^n}{RV_t^n} - \frac{[Y]_t^d}{[Y]_t}\right) \overset{fin.\ stably\ in\ law}{\longrightarrow} S_t^{(2)}, \tag{13}$$

where $S_t^{(2)}$ has, conditionally on $\mathcal{A}$, Gaussian law with zero mean and variance given by

$$\left(2\left(\frac{[Y]_t^c}{[Y]_t^2}\right)^2 - 4\frac{[Y]_t^c}{[Y]_t^3} + \frac{(2+\theta_I)}{[Y]_t^2}\right)\int_0^t \sigma_u^4 du + 4\left(\frac{[Y]_t^c}{[Y]_t^2}\right)^2 \sum_{p:\ T_P \le t} \left(\Delta Y_{T_p}\right)^2 \sigma_{T_p}^2. \tag{14}$$

Finally, we consider the logarithmic difference of the quadratic variation and the continuous part of the quadratic variation.

Corollary 4.5 *Assume (L-s) for some $s < 1$, (H') and that Y and σ have no common jumps. For $I \in \mathbb{N}$ with $2 < I < \frac{2}{s}(2-s)$, we obtain, as $n \to \infty$:*

$$\frac{1}{\sqrt{\Delta_n}} \left(\log\left(RV_t^n\right) - \log\left(RMPV(2;I)_t^n\right) - \left(\log\left([Y]_t\right) - \log\left([Y]_t^c\right)\right)\right) \overset{fin.\ stably\ in\ law}{\longrightarrow} S_t^{(3)}, \quad (15)$$

where $S_t^{(3)}$ has, conditionally on $\mathcal{A}$, Gaussian law with zero mean and variance given by

$$\left(\frac{2}{[Y]_t^2} - \frac{4}{[Y]_t[Y]_t^c} + \frac{(2+\theta_I)}{([Y]_t^c)^2}\right)\int_0^t \sigma_u^4 du + \frac{4}{[Y]_t^2}\sum_{0\le s\le t}\sigma_s^2(\Delta Y_s)^2. \quad (16)$$

Note that the difference, ratio or logarithmic difference of realised variance and realised multipower variation can be used as test statistics for testing for jumps in asset prices. Such tests have been developed by Barndorff-Nielsen & Shephard (2006). So, in order to test the null hypothesis that there are no jumps, it it sufficient to know the asymptotic distribution of the various test statistics under the null hypothesis. However, in order to make inference on the corresponding jump part of the quadratic variation (or the corresponding ratio or logarithmic difference), one has to know the asymptotic distribution of these test statistics under the alternative hypothesis, i.e. in the presence of jumps. So, Corollary 4.3 – 4.5 not only provide the tools for inference on the jump part of the quadratic variation, but also contain the asymptotic distributions of the well–known jump test statistics under the alternative distribution, which has not been known yet.

4.3 Feasible confidence bounds for the jump part of quadratic variation

The central limit theorems derived in the previous sections are infeasible in the sense that the asymptotic variances of the estimation bias are unobservable. In order to derive a feasible central limit theorem, we therefore need estimators for the asymptotic variances (12), (14), (16). From Barndorff-Nielsen & Shephard (2002) and Jacod (2006), we know that the continuous part of the asymptotic variance can be consistently estimated in the presence of jumps by special cases of the realised multipower variation. For integers $\tilde{I} \geq 3$, we get

$$RMPV(4;\tilde{I})_t^n \overset{ucp}{\longrightarrow} \int_0^t \sigma_s^4 ds, \quad \text{as } n \to \infty. \quad (17)$$

Note that $RMPV(4;\widetilde{I})$ for $\tilde{I} \leq 2$ tends to infinity in the presence of jumps, hence we choose $\tilde{I} \geq 3$.

So, how can we estimate the jump part of the asymptotic variance, which is (up to a constant) given by

$$\sum_{0\le s\le t}\left(\sigma_s^2 + \sigma_{s-}^2\right)\left(\Delta Y_s\right)^2\ ? \quad (18)$$

Aït-Sahalia & Jacod (2006) have derived a consistent estimator for $D(p) = \sum_{s\leq t}(\sigma_{s-}^2 + \sigma_s^2)|\Delta Y_s|^p$ when $p > 3$. However, we are interested in the case $p = 2$, which is not covered by their work.

It turns out that one can use an estimator which is based on a generalised, weighted version of realised multipower variation for estimating (18). This result is stated in the following theorem.

Theorem 4.6 *Let $Y = (Y_t)_{t\geq 0}$ be an Itô semimartingale satisfying assumption (L–s) for $0 \leq s < 2$. Let $(K_n)_{n\geq 0} \subset \mathbb{N}$ be any sequence which satisfies $K_n \to \infty$ and $K_n\Delta_n \to 0$ as $n \to \infty$. Further, we define an index set $I(n,i) = I^+(n,i) \cup I^-(n,i)$ where $I^-(n,i) = \{j \in \mathbb{N} \cap \{i - K_n + 1, \ldots, i-2\}\}$ and $I^+(n,i) = \{j \in \mathbb{N} \cap \{i+2, \ldots, i+K_n-1\}\}$ for $i = 1, \ldots, [t/\Delta_n]$. Let*

$$\begin{aligned} \widehat{\sigma}^2_{(i-1)\Delta_n -} &= \frac{\mu_1^{-2}}{(K_n-2)\Delta_n} \sum_{j\in I^-(n,i)} \left|\Delta_j^n Y\right| \left|\Delta_{j+1}^n Y\right|, \\ \widehat{\sigma}^2_{(i-1)\Delta_n +} &= \frac{\mu_1^{-2}}{(K_n-2)\Delta_n} \sum_{j\in I^+(n,i)} \left|\Delta_j^n Y\right| \left|\Delta_{j+1}^n Y\right|. \end{aligned} \tag{19}$$

Then for each $t \geq 0$

$$\widehat{J}_t^n = \sum_{i=1}^{[t/\Delta_n]} \left(\widehat{\sigma}^2_{(i-1)\Delta_n -} + \widehat{\sigma}^2_{(i-1)\Delta_n +}\right) \left(\Delta_i^n Y\right)^2 \xrightarrow{\mathbb{P}} J_t, \quad \text{as } n \to \infty, \tag{20}$$

where

$$J_t = 2\int_0^t \sigma_s^4 ds + \sum_{0\leq s\leq t} \left(\sigma_{s-}^2 + \sigma_s^2\right) \left(\Delta Y_s\right)^2.$$

Proof The proof is given in Appendix A. □

Remark Here we use locally averaged realised bipower variation as defined in (19) for estimating the spot variance. This estimator has been proposed by Lee & Mykland (2006) when Y is a jump diffusion. So our result extends their result by showing that their estimator can also be used in the framework of an Itô semimartingale. Alternatively, one could estimate the spot variance by locally averaged truncated realised variance as proposed by Aït-Sahalia & Jacod (2006). However, we have compared the performance of these two spot variance estimators by means of Monte Carlo studies (whose exact details are not presented here). The simulation results suggested that the locally averaged realised bipower variation performs slightly better than truncated realised variance. This result might be due to the fact that it is not easy to choose a sensible threshold for the truncated realised variance.

From (17) and (20), we can now derive a general estimator for any linear combination of the jump part (18) and the integrated fourth power of the volatility process.

Corollary 4.7 *Assume the assumptions of Theorem 4.6 are satisfied. Let c_1, c_2 denote constants with $2c_1 \geq c_2$ and let $\tilde{I} \geq 3$ be an integer. Then, we obtain, as $n \to \infty$,*

$$c_1 \sum_{i=1}^{t/\Delta_n} \left(\hat{\sigma}^{2(-)}_{(i-1)\Delta_n} + \hat{\sigma}^{2(+)}_{(i-1)\Delta_n} \right) (\Delta_i^n Y)^2 - c_2 \frac{1}{\Delta_n} RMPV(4; \tilde{I})_t^n$$
$$\xrightarrow{\mathbb{P}} (2c_1 - c_2) \int_0^t \sigma_s^4 ds + c_1 \sum_{0 \leq s \leq t} \left(\sigma_{s-}^2 + \sigma_s^2 \right) (\Delta Y_s)^2. \quad (21)$$

However, the left hand side of (21) can become negative in finite samples. Hence, in order to make sure that the estimator for the variance is always positive, we make the following finite sample correction.

Corollary 4.8 *Under the assumptions of Theorem, 4.6 we define*

$$\hat{A}_t^n(c_1, c_2, \tilde{I}) = \max \left\{ c_1 \sum_{i=1}^{t/\Delta_n} \left(\hat{\sigma}^{2(-)}_{(i-1)\Delta_n} + \hat{\sigma}^{2(+)}_{(i-1)\Delta_n} \right) (\Delta_i^n Y)^2 - c_2 \frac{1}{\Delta_n} RMPV(4; \tilde{I})_t^n, \right.$$
$$\left. (2c_1 - c_2) \frac{1}{\Delta_n} RMPV(4; \tilde{I})_t^n \right\},$$

for constants c_1, c_2 with $2c_1 \geq c_2$ and for an integer $\tilde{I} \geq 3$. Clearly, $\hat{A}_t^n(c_1, c_2)$ converges to the right–hand side of (21), as $n \to \infty$ and, hence, we obtain:

$$\hat{A}_t^n(2, 4 - \theta_I, \tilde{I}) \to \theta_I \int_0^t \sigma_s^4 ds + 2 \sum_{0 \leq s \leq t} \left(\sigma_{s-}^2 + \sigma_s^2 \right) (\Delta Y_s)^2 .$$

Remark Clearly, in the absence of common jumps of Y and σ, we could also use the slightly simpler estimator of the asymptotic variance given by

$$\max \left\{ 2c_1 \sum_{i=1}^{t/\Delta_n} \hat{\sigma}^{2(-)}_{(i-1)\Delta_n} (\Delta_i^n Y)^2 - c_2 \frac{1}{\Delta_n} RMPV(4; \tilde{I})_t^n, (2c_1 - c_2) \frac{1}{\Delta_n} RMPV(4; \tilde{I})_t^n \right\}.$$

Now we can derive a feasible asymptotic result for the difference of realised variance and realised multipower and can, eventually, make inference on the jump part of quadratic variation.

Corollary 4.9 *Assume that (L-s) holds for $s < 1$ and that (H') is satisfies. Furthermore, assume that Y and σ have no common jumps and let $\widetilde{I}, \widehat{I} \geq 3$ and $2 < I < \frac{2}{s}(2 - s)$ be integers.*

Let $\alpha \in [0, 1]$ and let $q_{1-\frac{\alpha}{2}}$ denote the $\left(1 - \frac{\alpha}{2}\right)$–quantile of the standard normal distribution. In the following, we state the asymptotic two–sided confidence bounds for the level $(1 - \alpha)$.

- *From Corollary 4.3, we obtain asymptotic confidence bounds for $[Y]_t^d$ given by*

$$(RV_t^n - RMPV(2; I)_t^n) \pm q_{1-\frac{\alpha}{2}} \sqrt{\Delta_n \hat{A}_t^n(4, 4 - \theta_I, \widetilde{I})}.$$

- *From Corollary 4.4, we obtain asymptotic confidence bounds for* $\frac{[Y]_t^d}{[Y]_t^n}$ *given by*

$$\left(1 - \frac{RMPV(2;I)_t^n}{RV_t^n}\right) \pm q_{1-\frac{\alpha}{2}} \sqrt{\Delta_n \hat{A}_t^n(c_1, c_2, \widetilde{I})},$$

where

$$c_1 = 2\left(\frac{RMPV(2,\widehat{I})_t^n}{(RV_t^n)^2}\right)^2,$$
$$c_2 = 2\left(\frac{RMPV(2,\widehat{I})_t^n}{(RV_t^n)^2}\right)^2 + 4\frac{RMPV(2,\widehat{I})_t^n}{(RV_t^n)^3} - \frac{(2+\theta_I)}{(RV_t^n)^2}.$$

- *From Corollary 4.5, we obtain asymptotic confidence bounds for*

$$\left(\log\left([Y]_t\right) - \log\left([Y]_t^c\right)\right),$$

given by

$$\left(\log\left(RV_t^n\right) - \log\left(RMPV(2;I)_t^n\right)\right) \pm q_{1-\frac{\alpha}{2}} \frac{\sqrt{\Delta_n \hat{A}_t^n(c_1, c_2, \widetilde{I})}}{RV_t^n RMPV(2;\widehat{I})_t^n},$$

where

$$c_1 = 4(RMPV(2;\widehat{I})_t^n)^2,$$
$$c_2 = 4(RMPV(2;\widehat{I})_t^n)^2 - (2(RMPV(2;\widehat{I})_t^n)^2 - 4RV_t^n RMPV(2;\widehat{I})_t^n$$
$$+ (2+\theta_I)(RV_t^n)^2).$$

4.4 Feasible confidence bounds for the entire quadratic variation and for integrated variance in the presence of jumps

For completeness, we state the result for making inference on continuous part and on the entire quadratic variation in the presence of jumps.

From Jacod (2007*a*, 2006) and Theorem 4.6, we obtain the following confidence bounds for the entire quadratic variation and for the integrated variance in the presence of jumps.

Corollary 4.10 *Assume that (L-s) holds for $0 \leq s < 2$. Furthermore, assume that Y and σ have no common jumps and let let $\widetilde{I} \geq 3$ be an integer. Let $\alpha \in [0,1]$ and let $q_{1-\frac{\alpha}{2}}$ denote the $(1-\frac{\alpha}{2})$–quantile of the standard normal distribution.*

- *The asymptotic $(1-\alpha)$–two–sided confidence bounds for $[Y]_t$ are given by*

$$RV_t^n \pm q_{1-\frac{\alpha}{2}} \sqrt{\Delta_n A_t^n(2, 2, \widetilde{I})}.$$

- *Additionally assume that (L-s) holds for $s < 1$ and $2 < I < \frac{2}{s}(2-s)$ for an integer I. The asymptotic $(1-\alpha)$–two–sided confidence bounds for $[Y]_t^c$ are given by*

$$RMPV(2,I)_t^n \pm q_{1-\frac{\alpha}{2}}\sqrt{\Delta_n(2+\theta_I)RMPV(4,\widetilde{I})_t^n}.$$

5 Simulation study

In this section, we will study the finite sample performance of the difference, ratio and logarithmic difference of realised variance and realised multipower variation. In the following, we will sometimes refer to theses quantities as linear, ratio and log–linear test statistic, respectively.

5.1 Simulation design

The simulation design for the Monte Carlo study is taken from Huang & Tauchen (2005) and the longer (unpublished) version of Barndorff-Nielsen, Shephard & Winkel (2006). We set $t = 1$ and simulate repeatedly over the time interval $[0, 1]$ a standard Brownian semimartingale $\int_0^t \sigma_s dW_s$ plus three different types of jump processes L, hence $Y_t = \int_0^t \sigma_s dW_s + L_t$. For the stochastic volatility process, we choose $\sigma_t = \exp(\beta_1 v_t)$, where $dv_t = -\lambda v_t dt + dB_t$ where B is a standard Brownian motion which is correlated with W and has correlation coefficient $\rho = -0.62$. Further, we set $\lambda = 0.1$ and $\beta_1 = 0.125$. Clearly, $\int \sigma dW$ and L are independent of one another. The jump process L will be chosen such that they have zero mean and unit unconditional variance, are symmetrical and share identical first four moments. We simulate from the various models by using the Euler scheme, where we choose an increment of one second per tick on the Euler clock. (i.e. $dt = 1/23400$ when we assume that the market is open from 9.30 am to 16.00 pm, which corresponds to 23400 seconds per day). The specifics of the jump processes are as follows:

BSM: $L \equiv 0$, i.e. the model consists only of the Brownian semimartingale part and has no jumps.

BSMSCP1: L is a stratified normal inverse Gaussian compound Poisson process with a single jump per unit time (i.e. the jump time is uniformly distributed over $[0, 1]$ and the jump size follows a normal inverse Gaussian distribution). The jump size is drawn from $\epsilon\sqrt{S}$ where $\epsilon \perp\!\!\!\perp S$, $\epsilon \sim N(0, 1)$ and $S \sim IG(c, c^2/\gamma)$. Then, $Var(\epsilon S) = \mathbb{E}(S) = c$ and $Var(S) = c^3/(c^2/\gamma) = c\gamma$. In the simulation, we set $\gamma = 0.25$.

BSMSCP10: L is a stratified normal inverse Gaussian compound Poisson process with ten jumps per unit time (i.e. the jump times are uniformly distributed over $[0, 1]$ and the jump size follows a normal inverse Gaussian distribution). Obviously, the jumps will have smaller variance than in BSMSCP1 in order to account for the fact that there are more jumps. Hence, the jump sizes are drawn from $\epsilon_i\sqrt{S_i}$ where $\epsilon_i \perp\!\!\!\perp S_i$, $\epsilon_i \sim N(0,1)$ and $S_i \sim IG(c/10, c^2/(10\gamma))$, $i = 1,\dots,10$. Then, $Var(\sum_{i=1}^{10}\epsilon_i\sqrt{S_i}) = \sum_{i=1}^{10}\mathbb{E}(S_i) = c$ and $Var(\sum_{i=1}^{10}S_i) = \frac{(\sum_{i=1}^{10}c/10)^3}{\sum_{i=1}^{10}c^2/(10\gamma)} = c^3/(c^2/\gamma) = c\gamma$. Again, we choose $\gamma = 0.25$.

BSMVG: $L \overset{L}{=} \epsilon\sqrt{S}$ is a variance Gamma process with $\epsilon \perp\!\!\!\perp S$, $\epsilon \sim N(0,1)$ and $S \sim \Gamma(c/\gamma, 1/\gamma)$, where $\gamma = 0.25$. Clearly, $L \overset{L}{=} \sum_{i=1}^{[1/dt]}\epsilon_i\sigma_i$, where $\epsilon_i \perp S_i$, $S_i \sim \Gamma(cdt/\gamma, 1/\gamma)$ and $\epsilon_i \sim N(0,1)$. $Var(\epsilon S) = \mathbb{E}(S) = c$ and $Var(S) = c/\gamma/(1/\gamma)^2 = c\gamma$.

So our simulation study deals with jump processes of both finite and infinite activity. Since the jump processes defined above all have Blumenthal-Getoor index 0, our theory holds for all $RMPV(2,I)$ with integers $I \geq 3$.

Note that $Var\left(\int_0^1 \sigma_s dW_s\right) = \mathbb{E}\left(\int_0^t \sigma_s^2 ds\right)$. Due to our parameter choice this variance is close to 1 in all the simulations carried out. Clearly, when we set $c = 1$, then the jump part J has also unit variance. However, empirical studies by Huang & Tauchen (2005) and Barndorff-Nielsen, Shephard & Winkel (2006) suggests that the jump component only accounts for up to 10 % of the variation of the price process, which corresponds to setting $c = 0.1$.

In the simulation experiment, we compute the feasible linear, ratio and log–linear statistics, where $[Y]_t^d$ is approximated by $\sum_{i=1}^{1/dt}\left(L_{idt} - L_{(i-1)dt}\right)^2$. Furthermore, we choose $\widetilde{I} = \widehat{I} = 10$ and, hence, use $RMPV(4,10)$ for estimating $\int_0^1 \sigma_s^4 ds$ and $RMPV(2,10)$ for estimating $\int_0^1 \sigma_s^2 ds$ (for the ratio and log–linear statistic). Recall, that we can choose $RMPV(4,\widetilde{I})$ for any integer $\widetilde{I} \geq 3$ for estimating integrated quarticity. However, by choosing a fairly big $\widetilde{I}$, we obtain better finite sample results. We simulate data for 5000 days and report the sample mean, standard deviation and the 95% coverage.

5.2 Efficiency and bias

So far, we have seen that we can use any realised multipower variation $RMPV(2;I)_t^n$ with $I \geq 3$ in our central limit theorems above as long as $\frac{s}{2-s} < 2/I < 1$ is satisfied. So, which multipower variation performs best in finite samples? In order to answer this question we compare the performance of the various test statistics based on four different power variations: tripower, quadpower, 10–power and 20–power. We know that in the absence of jumps, realised variance is the most efficient consistent estimator of integrated variance. Using higher multipower variation in such a model setting results in an efficiency loss, which can be quantified by θ_I. Recall that we have $\mu_r = \mathbb{E}|U|^r = \frac{\sqrt{2^r}\Gamma\left(\frac{1}{2}(r+1)\right)}{\sqrt{\pi}}$, for $U \sim N(0,1)$,

$$\theta_I = \left(\mu_{2/I}^{-I}\sqrt{\omega_I^2}\right)^2 - 2 = \mu_{2/I}^{-2I}\ \omega_I^2 - 2 \qquad \omega_I^2 = \mu_{4/I}^I + (1-2I)\mu_{2/I}^{2I} + 2\sum_{j=1}^{I-1}\mu_{4/I}^{I-j}\ \mu_{2/I}^{2j}.$$

It turns out that θ_I is an increasing function in I (some of its values are given in Table 2). Clearly, an increase in θ_I corresponds to a loss in efficiency when estimating the continuous part of the quadratic variation by $RMPV(2, I)$. However, the loss in efficiency is bounded by a constant since

$$\lim_{I\to\infty} \theta_I = \frac{\pi^2}{2} - 2.$$

I	1	2	3	4	5	6	∞
θ_I	0	0.608	1.061	1.377	1.605	1.776	2.934

Table 2: Different values for θ_I

Given these results, we might want to focus on realised tripower or quadpower variation for estimating integrated variance in the presence of jumps rather than using higher powers. However, our simulation study reveals the fact that realised multipower variation for higher powers (10 or 20) seems to outperform realised tri- and quadpower variation in the presence of jumps, since the estimation bias is much smaller compared to tri- and quadpower and the coverage is closer to the theoretical value of 95% for bigger I. This might be due to the fact that by using higher values of I, it is much easier to smooth out jumps and, hence, get a more reliable estimate of integrated variance in the presence of jumps.

5.3 Simulation results

Before we study the various jump processes, we check the performance of the various statistics when the log–price is just given by a standard Brownian semimartingale (BSM). The results of this study are given in Table 3. Generally speaking the results are good also for a low number $M = [1/\Delta_n]$ of intraday observations. But we also observe that 10–power and 20–power seem to perform slightly better for small M. This is somehow surprising, since in the absence of jumps, usually lower power variation performs better. However, we have to bare in mind that our test statistics incorporate also estimates of spot variances, which are not that reliable in small samples and can, hence, lead to deviations from the asymptotic normality as indicated by the slightly too high coverage for tri- and quadpower in finite samples.

Now we study the model of a Brownian semimartingale plus a stratified compound Poisson process with one jump per day (BSMSCP1). The results are given in Table 4. In the more relevant case that $c = 0.1$, we observe a generally good finite sample results for all three statistics. However, we also see that there seem to be a finite sample bias which converges to zero fairly slowly, but this bias seems to be much smaller for higher powers of I. For $c = 1$, the finite sample behaviour for $M = 39, 78$ is not really good, i.e. the finite sample bias is quite big and the coverage is far to low compared to 95%. However, from $M = 390$ (which corresponds to 1 min increments) onwards, the finite sample performance of 10- and 20–power is already really good, whereas it is still fairly weak for tri- and quadpower.

In the following we will only report the results for $c = 0.1$, which is the empirically relevant case. Next we consider a stratified compound Poisson process with 10 jumps per day (BSM-SCP10). The corresponding results are given in Table 5. Now it looks like we obtain the best finite sample results for realised 10–power. The coverage seems to be slightly closer to 95% for the linear and ratio statistic than for the log–linear statistic.

Finally we choose variance gamma process as jump component, which is still a process of Blumenthal–Getoor index 0 and, hence, of finite variation, but has infinite activity. The results which are given in Table 6 follow a similar pattern as the ones we obtained before and show in general a good finite sample performance.

6 Conclusion

In this paper, we have derived the joint distribution of realised variance and realised multipower in the presence of jumps. From this result, we have deduced the corresponding asymptotic theory for the difference, the ratio and the logarithmic difference of realised variance and realised multipower variation, which makes it possible to make inference on the jump part of the quadratic variation. Note that the difference, the ratio and the logarithmic difference are often used for testing for jumps in asset prices. So far, the asymptotic distribution of these test statistics has only been known under the null hypothesis that there are no jumps. Hence, this paper completes the picture by presenting the asymptotic distributions of these statistics under the alternative hypothesis that there are jumps. At first, the new central limit theorems turned out to be infeasible since the asymptotic variances of the estimation biases are not observable. However, we have introduced new, consistent estimators for these asymptotic variances which, eventually, leads to a feasible central limit theory. These new estimators can also be used for deriving a feasible central limit theory for making inference on the entire quadratic variation in the presence of jumps. Finally we have checked the finite sample performance of our asymptotic results by means of Monte Carlo studies. We have obtained particularly good results when we use high multipower variations (e.g. 10–power variation).

In future work, it will be interesting to study in particular two questions in more detail. First, how do the results change when we allow for market microstructure noise in the model? How robust are our test statistics and how does the asymptotic distribution change? Second, how do these results extend to a multivariate framework? Very recent work by Barndorff-Nielsen & Shephard (2007) and Jacod & Todorov (2007) has already addressed the question of testing for common and disjoint jumps of multivariate price processes. So it would be very interesting to see whether it would be possible to extend the results from this paper to a multivariate model setting.

A Proofs

A.1 Proof of the bivariate central limit theorem

Proof of Theorem 4.1 The univariate results follow from Jacod (2007*a*, Theorem 2.11 (ii)) and Jacod (2006, Theorem 6.2). In order to derive the multivariate central limit result, we use a modified version of Jacod (2007*a*, Theorem 2.12) which can account for multipower variation rather than power variation only. For the proof of the theorem, we essentially have to prove three lemmas (Lemma A.1 – Lemma A.3), which we will do in the following. For our proofs, we have to introduce some stronger assumptions (see e.g. Jacod (2007*a*)), which can be later relaxed to the ones stated in the theorems above.

Hypothesis (SH) The hypothesis (H) holds and the processes (b_t), (c_t) and $(F_t(\phi_2))$ are bounded by a non–random constant and the jumps of Y are also bounded by a constant.

We refer to Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006) and Jacod (2007*a*, Section 4 and 5) for more details about how the validity of the corresponding limit results under stronger hypothesis leads to their validity under (H).

Remark Barndorff-Nielsen, Graversen, Jacod & Shephard (2006, Theorem 2 (in particular, Example 7)) contains the following bivariate limit theorem for realised variance and realised bipower variation for a Brownian semimartingale Y, i.e. in the absence of jumps, as $n \to \infty$,

$$\frac{1}{\sqrt{\Delta_n}} \begin{pmatrix} RV_t^n - [Y]_t \\ RMPV((1,1))_t^n - [Y]_t \end{pmatrix} \xrightarrow{stably\ in\ law} \begin{pmatrix} \sqrt{2}\int_0^t \sigma_u^2 d\overline{W}_u \\ \sqrt{2}\int_0^t \sigma_u^2 d\overline{W}_u + \sqrt{\theta_2}\int_0^t \sigma_u^2 d\widetilde{W}_u \end{pmatrix},$$

stably in law for independent Brownian motions $\overline{W}$ and $\widetilde{W}$.

By using exactly the same reasoning, we can show that in the continuous semimartingale framework we obtain, as $n \to \infty$,

$$\frac{1}{\sqrt{\Delta_n}} \begin{pmatrix} RV_t^n - [Y]_t \\ RMPV(2;I)_t^n - [Y]_t \end{pmatrix} \xrightarrow{stably\ in\ law} \begin{pmatrix} \sqrt{2}\int_0^t \sigma_u^2 d\overline{W}_u \\ \sqrt{2}\int_0^t \sigma_u^2 d\overline{W}_u + \sqrt{\theta_I}\int_0^t \sigma_u^2 d\widetilde{W}_u \end{pmatrix},$$

stably in law for independent Brownian motions $\overline{W}$ and $\widetilde{W}$.

In order to prove our main theorem, we need some further notation, which we will introduce in the following. Let Y_t denote a one–dimensional Itô semimartingale. We consider functions $g_j : \mathbb{R} \to \mathcal{M}_{d_j,d_{j+1}}$ for $j = 1, \ldots, I$, where $\mathcal{M}_{d_j,d_{j+1}}$ denotes a $d_j \times d_{j+1}$–dimensional matrix with real–valued entries. Note that we are in particular interested in the following choice of functions g_j for $j = 1, \ldots, I$ and $I \geq 3$. Let $d_1 = \cdots = d_I = 2$, $d_{I+1} = 1$ and

$$g_1(y) = \begin{pmatrix} y^2 & 0 \\ 0 & \mu_{2/I}^{-1}|y|^{2/I} \end{pmatrix}, \quad g_i(y) = \begin{pmatrix} 1 & 0 \\ 0 & \mu_{2/I}^{-1}|y|^{2/I} \end{pmatrix}, \quad g_I(y) = \begin{pmatrix} 1 \\ \mu_{2/I}^{-1}|y|^{2/I} \end{pmatrix}, \quad (22)$$

for $i = 2, \ldots I-1$. Then,

$$\Delta_n \sum_{i=1}^{[t/\Delta_n]-I+1} \prod_{i'=1}^{I} g_{i'} \left(\frac{\Delta_{i+i'-1}^n Y}{\sqrt{\Delta_n}} \right) = \left(\begin{array}{c} RV_t^n - \sum_{i=[t/\Delta_n]-I}^{[t/\Delta_n]} (\Delta_i^n Y)^2 \\ RMPV(2;I)_t^n \end{array} \right),$$

where $\sum_{i=[t/\Delta_n]-I}^{[t/\Delta_n]} (\Delta_i^n Y)^2 \stackrel{ucp}{\longrightarrow} 0$ as $n \to \infty$. Further, we define $\beta_i^{i'} = \frac{1}{\sqrt{\Delta_n}} \sigma_{(i-1)\Delta_n} \Delta_{i+i'-1}^n W$ for $i' = 1, \ldots, I$ and

$$\rho_i^n(g_{i'}) = \int g_{i'}(x) f_{\sigma_{(i-1)\Delta_n}}(x) dx,$$

(componentwise for the diagonal matrices and vectors defined above), where $f_{\sigma_{(i-1)\Delta_n}}$ is the density of a $N\left(0,\ \sigma_{(i-1)\Delta_n}^2\right)$–distributed random variable. So, finally, we define the following random vector:

$$\overline{U}_t^n = \overline{U}^n(g_1, \ldots, g_I)_t = \sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]-I+1} \left\{ \prod_{i'=1}^{I} g_{i'}\left(\beta_i^{i'}\right) - \prod_{i'=1}^{I} \rho_i^n\left(g_{i'}\right) \right\}, \qquad (23)$$

which is in $\mathcal{M}_{d_1, d_{I+1}}$.

Lemma A.1 *Assume that (SH) holds and let $g_1, \ldots, g_I$ denote continuous even functions of at most polynomial growth with $g_i : \mathbb{R}^d \to \mathcal{M}_{d_i, d_{i+1}}$ for $i = 1, \ldots I$ as defined in (22). So, in particular, we have $d_1 = 2$ and $d_{I+1} = 1$. Let $\overline{U}^n = \overline{U}^n(g_1, \ldots g_I)$ denote the stochastic process defined in (23) with components*

$$\overline{U}_t^n (g_1, \ldots, g_I)^{(j)} = \sqrt{\Delta_n} \sum_{i=1}^{[t/\Delta_n]-I+1} \left\{ \left(\prod_{i'=1}^{I} g_{i'}\left(\beta_i^{i'}\right) \right)^{(j)} - \left(\prod_{i'=1}^{I} \rho_i^n\left(g_{i'}\right) \right)^{(j)} \right\},$$

for $j = 1, 2$. Then the d_1–dimensional process $\overline{U}^n$ converges stably in law to a limit process $\overline{U}$ with components

$$\overline{U}_t^{(j)} = \sum_{k=1}^{2} \int_0^t \Sigma_u^{j,k} d\overline{W}_u^k, \quad j = 1, 2,$$

where the 2×2–dimensional process Σ, defined by

$$\Sigma_u = \left(\begin{array}{cc} \sqrt{2}\sigma_u^2 & 0 \\ \sqrt{2}\sigma_u^2 & \sqrt{\theta_I}\sigma_u^2 \end{array} \right) \qquad (24)$$

is $(\mathcal{F}_t)$–optional.

Proof Since we are only dealing with Brownian semimartingales in this lemma, the result follows directly along the lines of Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006, Proposition 5.2) or by extending the proof of Jacod (2007*a*, Lemma 5.7), which we will sketch in the following.

One can easily show by induction on I that

$$\prod_{i'=1}^{I} g_{i'}\left(\beta_i^{i'}\right) - \prod_{i'=1}^{I} \rho_i^n(g_{i'}) = \sum_{j=1}^{I}\left(\prod_{i'=1}^{j-1} g_{i'}\left(\beta_i^{i'}\right)\right)\left[g_j\left(\beta_i^j\right) - \rho_i^n(g_j)\right]\left(\prod_{i'=j+1}^{I} \rho_i^n(g_{i'})\right),$$

where an empty product is set to 1. This term is not measurable with respect to $\mathcal{F}_{i\Delta_n}$, which we need in order to be able to apply Jacod & Shiryaev (2003, Theorem IX.7.19 and Theorem IX.7.28). So we use the same methods which have been applied in the proof of Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006, Proposition 5.2). I.e. we shift the terms back in time to make them measurable w.r.t. $\mathcal{F}_{i\Delta_n}$. We do not shift the first term in the sum, but we shift the second term by one, the third term by two etc. and, finally, the Ith term by $I-1$. By doing that we get a new random variable

$$\zeta_i^n = \sqrt{\Delta_n}\sum_{j=1}^{I}\left(\prod_{i'=1}^{j-1} g_{i'}\left(\beta_i^{i'-(j-1)}\right)\right)\left[g_j\left(\beta_i^1\right) - \rho_{i-(j-1)}^n(g_j)\right]\left(\prod_{i'=j+1}^{I} \rho_{i-(j-1)}^n(g_{i'})\right), \quad (25)$$

which is clearly measurable with respect to $\mathcal{F}_{i\Delta_n}$. As in the proof of Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006, Proposition 5.2) one can easily show that

$$\overline{U}_t^n(g_1,\ldots,g_I) - \sum_{i=I}^{[t/\Delta_n]-I+1} \zeta_i^n \xrightarrow{ucp} 0, \quad \text{as } n\to\infty.$$

Let $\mathbb{E}_{i-1}^n(\cdot) = \mathbb{E}\left(\cdot|\mathcal{F}_{(i-1)\Delta_n}\right)$. Trivially, we get $\mathbb{E}_{i-1}^n\left(\zeta_i^n\right) = 0$ and $\mathbb{E}_{i-1}^n\left(||\zeta_i^n||^4\right) \le K\Delta_n^2$ (for a constant $K>0$). Analogously to the proof of Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006, Proposition 5.2), we obtain in particular that

$$\sum_{i=1}^{[t/\Delta_n]-I+1} \mathbb{E}_{i-1}^n\left(\zeta_i^{j;n}\zeta_i^{k;n}\right) \xrightarrow{ucp} \int_0^t \left(\Sigma_u\Sigma_u^*\right)^{j,k} du, \quad (26)$$

$$\sum_{i=1}^{[t/\Delta_n]-I+1} \mathbb{E}_{i-1}^n\left(\zeta_i^{j;n}\Delta_i^n N\right) \xrightarrow{ucp} 0, \quad \text{if } N=W \text{ or } N\in\mathcal{N}, \quad (27)$$

as $n\to\infty$, where $\mathcal{N}$ is the set of all bounded $(\mathcal{F}_t)$–martingales which are orthogonal to W. Now the result follows from Jacod & Shiryaev (2003, Theorem IX.7.19 and Theorem IX.7.28). □

Now we study the more general case where we allow for jumps in the price process Y. We start by introducing some notation (which is the same as in Jacod (2007*a*)) and some more assumptions, which can be relaxed later.

Hypothesis (SK) Assumptions (K) and (SH) are satisfied and the functions $\gamma_k = \gamma$ are bounded and do not depend on k.

Let $\epsilon > 0$ fixed. We define a process N by $N = \mathbb{1}_E * \underline{\mu}$, where $E = \{x : \gamma(x) > \epsilon\}$. Hence N is a Poisson process with parameter the Lebesgue measure of E, say λ.

Remark Note that under (SK) we have $\int_{\mathbb{R}}(1 \wedge \gamma^2(x))dx < \infty$ and $\sup_x \gamma(x) \leq K$ for a $K \geq 0$. Therefore we get $\int_{\mathbb{R}} \gamma^2(x)\mathbb{1}_{\{\gamma^2(x)\leq 1\}}(x)dx < \infty$ and $\int_{\mathbb{R}} \mathbb{1}_{\{\gamma^2(x)>1\}}(x)dx < \infty$. So altogether, we obtain $\int_{\mathbb{R}} \gamma^2(x)dx < \infty$, since

$$\begin{aligned}\int_{\mathbb{R}} \gamma^2(x)dx &= \int_{\mathbb{R}} \gamma^2(x)\mathbb{1}_{\{\gamma^2(x)\leq 1\}}(x)dx + \int_{\mathbb{R}} \gamma^2(x)\mathbb{1}_{\{\gamma^2(x)>1\}}(x)dx \\ &\leq \int_{\mathbb{R}} \gamma^2(x)\mathbb{1}_{\{\gamma^2(x)\leq 1\}}(x)dx + K\int_{\mathbb{R}} \mathbb{1}_{\{\gamma^2(x)>1\}}(x)dx < \infty.\end{aligned}$$

Therefore, we can deduce that λ is indeed finite:

$$\lambda = \int_{\mathbb{R}} \mathbb{1}_{\{x:\gamma(x)>\epsilon\}}(x)dx \leq \int_{\mathbb{R}} \frac{\gamma(x)^2}{\epsilon^2}dx < \infty.$$

Depending on ϵ, we define the following quantities:

- $S_1, S_2, \ldots$ are the successive jump times of N,
- $I(n,p) = i,\ S_-(n,p) = (i-1)\Delta_n,\ S_+(n,p) = i\Delta_n$ on $\{(i-1)\Delta_n < S_p \leq i\Delta_n\}$,
- $\alpha_-(n,p) = \frac{1}{\sqrt{\Delta_n}}\left(W_{S_p} - W_{S_-(n,p)}\right),\ \alpha_+(n,p) = \frac{1}{\sqrt{\Delta_n}}\left(W_{S_+(n,p)} - W_{S_p}\right)$,
- $R_p = \Delta Y_{S_p}$,
- $Y(\epsilon)_t = Y_t - \sum_{p:S_p\leq t} R_p$,
- $R_p'^n = \Delta_i^n Y(\epsilon)$ on the set $\{(i-1)\Delta_n < S_p \leq i\Delta_n\}$,
- $R_p' = \sqrt{\xi_p}U_p\sigma_{S_p-} + \sqrt{1-\xi_p}U_p'\sigma_{S_p}$,
- $\Omega_n(T,\epsilon) = \{\omega :$ each interval $[0,T] \cap ((i-1)\Delta_n, i\Delta_n]$ contains at most one $S_p(\omega)$; $|\Delta_i^n Y(\epsilon)(\omega)| \leq 2\epsilon, \forall i \leq T/\Delta_n\}$.

Lemma A.2 *Under (SK), the sequences* $\left(\overline{U}^n, (\alpha_-(n,p), \alpha_+(n,p))_{p\geq 1}\right)$ *converge stably in law to*
$\left(\overline{U}, \left(\sqrt{\xi_p}U_p, \sqrt{1-\xi_p}U_p'\right)_{p\geq 1}\right)$ *as* $n \to \infty$.

Proof The proof of this lemma is similar to the corresponding proof by Jacod (2007*a*, p. 30–31). However, we have to make some adjustments to allow for multipower variation. We have to prove that for all bounded $\mathcal{A}$–measurable random variables Ψ and all bounded Lipschitz functions Φ on the Skorohod space of d–dimensional functions on $\mathbb{R}_+$ endowed with a distance for the Skorohod topology, and all $q \geq 1$ and all continuous bounded functions f_p on $\mathbb{R}^2$, and with $A_n = \prod_{p=1}^{q} f_p(\alpha_-(n,p), \alpha_+(n,p))$ then

$$\mathbb{E}\left(\Psi\Phi\left(\bar{U}^n\right)A_n\right) \to \tilde{\mathbb{E}}\left(\Psi\Phi(\bar{U})\right)\prod_{p=1}^{q}\tilde{\mathbb{E}}\left(f_p(\sqrt{\xi_p}U_p, \sqrt{1-\xi_p}U_p^{'})\right), \quad \text{as } n \to \infty. \tag{28}$$

Replacing Ψ by $\mathbb{E}\left(\Psi|\,\mathcal{G}\right)$ on both sides, it is sufficient to prove the limit result (28) for a Ψ which is measurable with respect to the separable σ–field $\mathcal{G}$ generated by both the measure $\underline{\mu}$ and the processes b, σ, W and Y.

Let $\underline{\mu}'$ and $\underline{\mu}''$ ($\underline{\nu}'$ and $\underline{\nu}''$, respectively) denote the restrictions of $\underline{\mu}$ (and $\underline{\nu}$, respectively) to $\mathbb{R}_+ \times E^c$ and to $\mathbb{R}_+ \times E$. Further, let $(\mathcal{F}')$ denote the smallest filtration containing $(\mathcal{F}_t)$ such that $\underline{\mu}''$ is $\mathcal{F}_0^{'}$-measurable. Clearly W is a Wiener process and $\underline{\mu}'$ is a Poisson random measure with compensator $\underline{\nu}'$ relative to $(\mathcal{F}_t)$, but also relative to $(\mathcal{F}_t^{'})$.

Now we define a set of intervals surrounding the jump times of the Poisson process N. Let $m \in \mathbb{N}$ be any positive integer, then we define $S_p^{m-} = (S_p - 1/m)^+$, $S_p^{m+} = S_p + 1/m$ and $B_m = \cup_{p\geq 1}(S_p^{m-}, S_p^{m+}]$. Since the indicator function $\mathbb{I}_{B_m}(\omega, t)$ is $\mathcal{F}_0^{'} \otimes \mathbb{R}_+$–measurable, we can define the stochastic integral $W(m)_t = \int_0^t \mathbb{I}_{B_m}(u)dW_u$. Now let $(\mathcal{F}_t^{'m})$ denote the smallest filtration containing $(\mathcal{F}_t^{'})$ such that $W(m)$ is $(\mathcal{F}_0^{'m})$–measurable. Further, we define the set $\Gamma_n(m,t) = \{i \in \mathbb{N} : i \leq [t/\Delta_n] - I + 1 \text{ and } B_m \cap ((i-1)\Delta_n, i\Delta_n] = \emptyset\}$. Similarly to Jacod (2007*a*), we define two bivariate processes $\overline{U}^{'n}(m)$, where we just sum over the integers which are not "close" to the jump times, and $\overline{U}(m)$, with components:

$$\overline{U}^{'n}(m)_t^j = \sqrt{\Delta_n}\sum_{i\in\Gamma_n(m,t)}\left(\left(\prod_{i'=1}^{I} g_{i'}\left(\beta_i^{i'}\right)\right)^j - \left(\prod_{i'=1}^{I}\rho_i^n\left(g_{i'}\right)\right)^j\right),$$

$$\overline{U}(m)_t^j = \sum_{j'=1}^{2}\int_0^t \Sigma_u^{j,j'}\mathbb{I}_{B_m^c(u)}d\overline{W}_u^{j'},$$

where Σ is defined by (24) and $j = 1, 2$. Once again, note that both integrals are well–defined since $\overline{W}$ is a Brownian motion w.r.t. the smallest filtration containing $(\mathcal{F}_t^{\prime})$ and $\mathcal{F}_0^{'m}$ at time 0. Clearly, $B_m \to \cup_p\{S_p\}$ for $m \to \infty$ and, hence, $\overline{U}(m) \xrightarrow{ucp} \overline{U}$ as $m \to \infty$.

Note that

$$\Gamma^n(m,t)^c = \{i : i \leq [t/\Delta_n] - I + 1, B_m \cap ((i-1)\Delta_n, i\Delta_n] \neq \emptyset\}$$
$$\subseteq \left\{i : i \leq [t/\Delta_n] - I + 1, \exists p : |i\Delta_n - S_p| \leq \frac{2}{m}\right\}.$$

Note that in the following, the constant K can change from line to line, but will not depend on n, t and m (but will depend on ϵ).

Since the conditional expectation of $\zeta_i^{j;n}$ is zero, if we condition on the past before $(i-1)\Delta_n$

and the sequence of stopping times S_p, which are independent of W, i.e. $\mathbb{E}\left(\zeta_i^{j;n}\middle|\,\mathcal{F}'_{(i-1)\Delta_n}\right) = 0$, we reach that $\bar{U}^n(g_1,\ldots,g_I)_s^j - \bar{U}'^n(m)_s^j$ is indeed a martingale with respect to $(\mathcal{F}_t'^m)$. By applying Doob's inequality, we obtain the following:

$$\mathbb{E}\left(\sup_{s\le t}\left|\bar{U}^n(g_1,\ldots,g_I)_s^j - \bar{U}'^n(m)_s^j\right|^2\right) \le 4\Delta_n\,\mathbb{E}\left(\sum_{p=1}^{\infty}\sum_{i=1}^{[t/\Delta_n]-I+1}|\zeta_i^{j;n}|^2\mathbb{1}_{\{|i\Delta_n - S_p|\le 2/m\}}\right).$$

Since all functions g_i (for $i = 1,\ldots I$) are of at most polynomial growth, there exist constants $\tilde{p}_1,\ldots,\tilde{p}_I$ such that (by induction on I)

$$\mathbb{E}\left(\sup_{s\le t}\left|\bar{U}^n(g_1,\ldots,g_I)_s^j - \bar{U}'^n(m)_s^j\right|^2\right) \le K\Delta_n\mathbb{E}\left(\sum_{p=1}^{\infty}\sum_{1\le i\le[t/\Delta_n]-I+1:\exists p:|i\Delta_n - S_p|\le 2/m}\prod_{i'=1}^{I}(1+|\beta_i^{i',n}|^{\tilde{p}_{i'}})\right).$$

Since σ is bounded and for fixed p, we get

$$\#\left\{i : i\le[t/\Delta_n]-I+1, |i\Delta_n - S_p|\le\frac{2}{m}\right\}\le\frac{4}{m\Delta_n},$$

we obtain from (SH) that

$$\begin{aligned}\mathbb{E}\left(\sup_{s\le t}\left|\bar{U}^n(g_1,\ldots,g_I)_s^j - \bar{U}'^n(m)_s^j\right|^2\right) &\le \frac{K}{m}\mathbb{E}\left(\sum_{p=1}^{\infty}\mathbb{1}_{\{S_p\le t+1\}}\right)\\ &= \frac{K}{m}\sum_{p=1}^{\infty}\mathbb{P}\left(S_p\le t+1\right) = \frac{K}{m}\sum_{p=1}^{\infty}\mathbb{P}\left(N_{t+1}\ge p\right)\\ &= \frac{K}{m}\lambda(t+1).\end{aligned}$$

It is now sufficient to prove that for each m and for each $\mathcal{G}$–measurable and bounded Ψ for fixed m, as $n\to\infty$:

$$\mathbb{E}\left(\Psi\Phi\left(\bar{U}'^n(m)\right)A_n\right)\to\tilde{\mathbb{E}}\left(\Psi\Phi(\bar{U}(m))\prod_{p=1}^{q}\tilde{\mathbb{E}}\left(f_p(\sqrt{\xi_p}U_p,\sqrt{1-\xi_p}U'_p)\right)\right), \tag{29}$$

since Ψ is Lipschitz and bounded.

The remaining part of the proof is then identical to Step 3 and Step 4 in Jacod (2007*a*, p. 31) and is, hence, not given here. □

Finally, we generalise the results from Lemma A.2 and obtain the final auxiliary limit result which we need for the proof of our main theorem.

Lemma A.3 *Under the assumptions of Lemma A.2, the sequences* $\left(\overline{U}^n, \left(R_p'^n/\sqrt{\Delta_n}\right)_{p\geq 1}\right)$ *converge stably in law to* $\left(\overline{U}, \left(R_p'\right)_{p\geq 1}\right)$*, as* $n \to \infty$.

Proof This proof goes along the lines of the proof of Jacod (2007*a*, Lemma 5.9) and is, hence, omitted. □

Now we can combine the results from the three Lemmas above to deduce the result of Theorem 4.1 analogously to the proof of Jacod (2007*a*, Theorem 2.12). I.e. note that Lemma A.1 is multidimensional. The one–dimensional results have been deduced from the corresponding components of Lemma A.1 by Jacod (2007*a*, Theorem 2.11 (ii)) for the realised variance and for the realised multipower variation by Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006, p. 10–11) in the absence of jumps and Jacod (2006, Theorem 6.2) in the presence of jumps. So the way how these results are deduced from Lemma A.1 can be carried over separately for each component in the multidimensional case. So Theorem 4.1 holds. □

A.2 Proof of the consistency of the estimator of the asymptotic variance

In order to prove Theorem 4.6, we prove a couple of lemmas first. In the following, we will always use the following notation and assumptions. We have (SH) and we use the following notation: $Y_t = Y_t' + Y_t''$, where

$$Y_t' = Y_0 + \int_0^t b_s ds + \int_0^t \sigma_{s-} dW_s, \qquad Y_t'' = \kappa \star (\mu - \nu)_t + \kappa' \star \mu_t.$$

Further, we need some notation for the increments of the various parts of Y. Let

$$\beta_i^n = \frac{1}{\sqrt{\Delta_n}} \sigma_{(i-1)\Delta_n} \Delta_i^n W, \qquad \delta_i^n = \sqrt{\Delta_n} \beta_i^n, \qquad \rho_i^n = \rho_{\sigma_{(i-1)\Delta_n}},$$

$$\chi_i'^n = \frac{1}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} \left(b_s ds + (\sigma_{s-} - \sigma_{(i-1)\Delta_n}) dW_s\right), \quad \chi_i''^n = \frac{1}{\sqrt{\Delta_n}} \Delta_i^n Y'', \quad \chi_i^n = \chi_i'^n + \chi_i''^n.$$

So, in particular, we get

$$\Delta_i^n Y = \Delta_i^n Y' + \Delta_i^n Y'' = \sqrt{\Delta_n} \left(\beta_i^n + \chi_i'^n + \chi_i''^n\right) = \sqrt{\Delta_n} \left(\beta_i^n + \chi_i^n\right).$$

For the conditional expectation, we write $\mathbb{E}_{i-1}^n(\cdot) = \mathbb{E}\left(\cdot | \mathcal{F}_{(i-1)\Delta_n}\right)$. Clearly, the following inequalities hold under (SH) (see e.g. Jacod (2007*b*)) for $q > 0$ and for various constants

$K_q,\ K > 0:$

$$\mathbb{E}^n_{i-1}\left(|\beta^n_i|^q\right) \le K_q, \quad \mathbb{E}^n_{i-1}\left(|\Delta^n_i Y^c|^q\right) \le K_q \Delta_n^{q/2},$$
$$\mathbb{E}^n_{i-1}\left(|\chi_i^{\prime n}|^q\right) \le K_q \Delta_n^{1\wedge q/2}, \quad \mathbb{E}\left(|\sigma_{t+s}-\sigma_t|^q|\mathcal{F}_t\right) \le K_q s^{1\wedge q/2}, \tag{30}$$
$$\mathbb{E}^n_{i-1}\left(|\chi_i^{\prime\prime n}|\right) \le K.$$

If we additionally assume (L–s) for $s < 2$, we get a stronger inequality for the jump part (see e.g. Todorov & Bollerslev (2007)). I.e. in that case we can find an $s < \alpha < 2$ such that

$$\mathbb{E}^n_{i-1}\left(|\chi_i^{\prime\prime n}|\right) \le \begin{cases} K\sqrt{\Delta_n}, & \text{under (L–}s\text{) for } 0 \le s \le 1 \\ K\Delta_n^{1/\alpha-1/2}, & \text{under (L–}s\text{) for } 1 < s < \alpha < 2 \end{cases}. \tag{31}$$

Furthermore, we will use that

$$\mathbb{E}^n_{i-1}\left(|\Delta^n_i X''| \wedge \eta^2\right) \le K\Delta_n \left(\frac{\eta^2+\Delta_n}{\theta^2} + \Gamma(\theta)\right), \tag{32}$$

for all $\eta > 0$ and $\theta \in (0,1)$ with $\Gamma(\theta) \to 0$ as $\theta \to 0$ as shown in Aït-Sahalia & Jacod (2006).

Remark In the following, we will always work with a constant $K > 0$ which can change from line to line in the various proofs.

Recall that we denote by $(K_n)_{n\ge 0} \subset \mathbb{N}$ a sequence which satisfies $K_n \to \infty$ and $K_n\Delta_n \to 0$ as $n \to \infty$.

Lemma A.4 *Assume (SH). Let* $\widehat{J}_t^{\prime\prime n} = \sum_{i=1}^{[t/\Delta_n]} (\delta_i^n)^2 \frac{\mu_1^{-2}}{(K_n-2)\Delta_n} \sum_{j\in I(n,i)} |\delta^n_j||\delta^n_{j+1}|$*. Then* $\widehat{J}_t^{\prime\prime n} \xrightarrow{\mathbb{P}} 2\int_0^t \sigma_s^4 ds$ *as* $n \to \infty$.

Proof of Lemma A.4: First of all note that

$$\frac{\mu_1^{-2}}{(K_n-2)\Delta_n} \sum_{j\in I(n,i)} |\delta^n_j||\delta^n_{j+1}| = \frac{\mu_1^{-2}}{(K_n-2)} \sum_{j\in I(n,i)} |\beta^n_j||\beta^n_{j+1}| \xrightarrow{\mathbb{P}} \sigma^2_{T_q} + \sigma^2_{T_q-}.$$

This result is proved in Lemma A.9, where the notation is also explained.

So from the consistency of the estimators we get $\widehat{\sigma}^2_{(i-1)\Delta_n-} = \sigma^2_{(i-1)\Delta_n-} + o_{\mathbb{P}}(1)$ and $\widehat{\sigma}^2_{(i-1)\Delta_n+} = \sigma^2_{(i-1)\Delta_n} + o_{\mathbb{P}}(1)$ (uniformly in i) by using methods from Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006). So from Protter (2004, Chapter II.6, Theorem 30) we get

$$\widehat{J}_t^{\prime\prime n} = \sum_{i=1}^{[t/\Delta_n]} (\delta_i^n)^2 \frac{\mu_1^{-2}}{(K_n-2)\Delta_n} \sum_{j\in I(n,i)} |\delta^n_j||\delta^n_{j+1}| = \sum_{i=1}^{[t/\Delta_n]} (\delta_i^n)^2 \left(\widehat{\sigma}^2_{(i-1)\Delta_n-} + \widehat{\sigma}^2_{(i-1)\Delta_n+}\right)$$
$$= \sum_{i=1}^{[t/\Delta_n]} (\delta_i^n)^2 \left(\sigma^2_{(i-1)\Delta_n-} + \sigma^2_{(i-1)\Delta_n+} + o_{\mathbb{P}}(1)\right) \xrightarrow{\mathbb{P}} 2\int_0^t \sigma_s^4 ds.$$

□

Lemma A.5 *Under (SH), let* $\widehat{J}_t'''^n = \sum_{i=1}^{[t/\Delta_n]} (\delta_i^n)^2 \frac{\mu_1^{-2}}{(K_n-2)\Delta_n} \sum_{j\in I(n,i)} |\Delta_j^n Y'||\Delta_{j+1}^n Y'|$. *Then* $\widehat{J}_t'''^n - \widehat{J}_t''^n \xrightarrow{\mathbb{P}} 0$, *as* $n \to \infty$.

Proof of Lemma A.5: Note that

$$\widehat{J}_t'''^n - \widehat{J}_t''^n = \frac{\mu_1^{-2}}{(K_n-2)\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \sum_{j\in I(n,i)} (\delta_i^n)^2 \left\{ |\Delta_j^n Y'||\Delta_{j+1}^n Y'| - |\delta_j^n||\delta_{j+1}^n| \right\}.$$

Now we can write

$$|\Delta_j^n Y'||\Delta_{j+1}^n Y'| \; - \; |\delta_j^n||\delta_{j+1}^n| \;\; = \;\; |\Delta_j^n Y'| \left(|\Delta_{j+1}^n Y'| - |\delta_{j+1}^n|\right) \; + \; |\delta_{j+1}^n| \left(|\Delta_j^n Y'| - |\delta_j^n|\right).$$

From (30), we get

$$\begin{aligned}
&\mathbb{E}_{j-1}^n |\Delta_j^n Y'| \le K\sqrt{\Delta_n}, \qquad\qquad \mathbb{E}_{j-1}^n |\delta_j^n| \le K\sqrt{\Delta_n},\\
&\mathbb{E}_{j-1}^n \big||\Delta_j^n Y'| - |\delta_j^n|\big| \le \mathbb{E}_{j-1}^n |\Delta_j^n Y' - \delta_j^n| = \mathbb{E}_{j-1}^n |\sqrt{\Delta_n}\chi_j'^n| \le K\Delta_n,\\
&\mathbb{E}_{i-1}^n (\delta_i^n)^2 \le K\Delta_n.
\end{aligned}$$

By taking successively conditional expectations, we obtain

$$\mathbb{E}\left|\widehat{J}_t'''^n - \widehat{J}_t''^n\right| \le \frac{\mu_1^{-2}}{(K_n-2)\Delta_n} \frac{t\, 2(K_n-2)}{\Delta_n} K\Delta_n^2\sqrt{\Delta_n} = Kt\sqrt{\Delta_n} \to 0, \;\; \text{as} \;\; n\to\infty,$$

which completes the proof. □

Lemma A.6 *Under (SH), let* $\widehat{J}_t''''^n = \sum_{i=1}^{[t/\Delta_n]} (\Delta_i^n Y')^2 \frac{\mu_1^{-2}}{(K_n-2)\Delta_n} \sum_{j\in I(n,i)} |\Delta_j^n Y'||\Delta_{j+1}^n Y'|$. *Then* $\widehat{J}_t''''^n - \widehat{J}_t'''^n \xrightarrow{\mathbb{P}} 0$, *as* $n \to \infty$.

Proof of Lemma A.6: We write

$$\widehat{J}_t''''^n - \widehat{J}_t'''^n = \frac{\mu_1^{-2}}{(K_n-2)\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \sum_{j\in I(n,i)} \left((\Delta_i^n Y')^2 - (\delta_i^n)^2\right) |\Delta_j^n Y'||\Delta_{j+1}^n Y'|.$$

Note that

$$\begin{aligned}
|\,(\Delta_i^n Y')^2 - (\delta_i^n)^2\,| &= \left|2\sqrt{\Delta_n}\chi_i'^n\delta_i^n + \Delta_n\left(\chi_i'^n\right)^2\right| \le 2\sqrt{\Delta_n}\left|\chi_i'^n\right||\delta_i^n| + \Delta_n\left|\left(\chi_i'^n\right)^2\right|\\
&\le 2\sqrt{\Delta_n}\,\frac{1}{2}\left((\delta_i^n)^2 + \left(\chi_i'^n\right)^2\right) + \Delta_n\left|\left(\chi_i'^n\right)^2\right|.
\end{aligned}$$

Again we apply the results from (30) and we obtain

$$\mathbb{E}_{i-1}^{n} |\, (\Delta_i^n Y')^2 - (\delta_i^n)^2 \,| \leq K\sqrt{\Delta_n}\Delta_n + \Delta_n^2.$$

So altogether, we get

$$\begin{aligned}\mathbb{E}\left|\widehat{J}_t^{\prime\prime\prime\prime n} - \widehat{J}_t^{\prime\prime\prime n}\right| &\leq \frac{K}{(K_n-2)\Delta_n}\frac{t}{\Delta_n}2(K_n-2)\left(\sqrt{\Delta_n}\Delta_n + \Delta_n^2\right)\Delta_n \\ &= Kt(\sqrt{\Delta_n}+\Delta_n) \to 0, \quad \text{as } n \to \infty,\end{aligned}$$

which implies the result. □

Lemma A.7 *Under (SH), let* $\widehat{J}_t^{\prime(1)n} = \sum_{i=1}^{[t/\Delta_n]} (\Delta_i^n Y')^2 \frac{\mu_1^{-2}}{(K_n-2)\Delta_n} \sum_{j\in I(n,i)} |\delta_j^n||\delta_{j+1}^n|$. *Then* $\widehat{J}_t^{\prime(1)n} - \widehat{J}_t^{\prime\prime\prime\prime n} \xrightarrow{\mathbb{P}} 0$, *as* $n \to \infty$.

Proof of Lemma A.7 This result is obvious from the proofs of Lemma A.4, Lemma A.5 and Lemma A.6. □

In the following, we study the jump part of Y.

Lemma A.8 *Under (SH), let* $\widehat{J}_t^{\prime(2)n} = \sum_{i=1}^{[t/\Delta_n]} (\Delta_i^n Y'')^2 \frac{\mu_1^{-2}}{(K_n-2)\Delta_n} \sum_{j\in I(n,i)} |\delta_j^n||\delta_{j+1}^n|$. *Then* $\widehat{J}_t^{\prime(2)n} \xrightarrow{\mathbb{P}} \sum_{0\leq s\leq t}(\sigma_{s-}^2 + \sigma_s^2)\,(\Delta Y_s)^2$, *as* $n \to \infty$.

Proof of Lemma A.8 The proof of this lemma follows essentially from Aït-Sahalia & Jacod (2006, Proof of Lemma 2) and Jacod (2007*b*, Proof of Lemma 6.9) and is, therefore, not presented here.

□

Now we present a lemma which says that locally averaged bipower variation is a consistent estimator of local variance. This result follows basically directly from Lee & Mykland (2006).

Lemma A.9 *Assume (SH). Let* $(T_q)_{q\geq 0}$ *with* $T_q = T_q(\epsilon)$ *denote the jump times of the Poisson process* $\mu([0,t]\times\{x : \gamma(x) > \epsilon/2\})$. *Furthermore we define* $i(n,q) = \inf\{i : i\Delta_n \geq T_q\}$ *and* $I'^{(-)}(n,q) = I^{-}(n,i(n,q))$ *and* $I'^{(+)}(n,q) = I^{+}(n,i(n,q))$. *Then we obtain for* $n \to \infty$*:*

$$\frac{\mu_1^{-2}}{(K_n-2)\Delta_n} \sum_{j\in I'^{(-)}(n,q)} |\delta_j^n||\delta_{j+1}^n| \xrightarrow{\mathbb{P}} \sigma_{T_q-}^2, \tag{33}$$

$$\frac{\mu_1^{-2}}{(K_n-2)\Delta_n} \sum_{j\in I'^{(+)}(n,q)} |\delta_j^n||\delta_{j+1}^n| \xrightarrow{\mathbb{P}} \sigma_{T_q}^2. \tag{34}$$

Proof of Lemma A.9 Here we only show (33). The corresponding result (34) for the right limit can be proven analogously. As in a similar proof in Aït-Sahalia & Jacod (2006), we define

$$s_q^n = \inf_{u\in[T_q-K_n\Delta_n,T_q)} \sigma_u^2, \qquad S_q^n = \sup_{u\in[T_q-K_n\Delta_n,T_q)} \sigma_u^2,$$

and

$$U_q^n = \frac{\mu_1^{-2}}{(K_n-2)\Delta_n} \sum_{j\in I'^{(-)}(n,q)} |\Delta_j^n W||\Delta_{j+1}^n W|.$$

From Lee & Mykland (2006), we immediately get that $U_q^n \xrightarrow{\mathbb{P}} 1$ as $n\to\infty$. Since, furthermore, $s_q^n U_q^n \le \frac{\mu_1^{-2}}{(K_n-2)\Delta_n}\sum_{j\in I'^{(-)}(n,q)} |\delta_j^n||\delta_{j+1}^n| \le S_q^n U_q^n$ almost surely, we obtain (33). □

Lemma A.10 *Under (SH), let* $\widehat{J}_t'^{(3)n} = \sum_{i=1}^{[t/\Delta_n]} (\Delta_i^n Y'\Delta_i^n Y'') \frac{\mu_1^{-2}}{(K_n-2)\Delta_n} \sum_{j\in I(n,i)} |\delta_j^n||\delta_{j+1}^n|$. *Then* $\widehat{J}_t'^{(3)n} \xrightarrow{\mathbb{P}} 0$, *as* $n\to\infty$.

Proof of Lemma A.10 Note that, since X' is a continuous process and X'' is a pure jump process, standard properties of the quadratic covariation process (see e.g. Protter (2004)) imply that $\sum_{i=1}^{[t/\Delta_n]} (\Delta_i^n Y'\Delta_i^n Y'') \xrightarrow{\mathbb{P}} [Y',Y''] = 0$, as $n\to\infty$. Now we can use a similar proof as for Lemma A.8 where we replace $\left(\Delta Y''_{T_q(\epsilon)}\right)^2$ by $\left(\Delta Y'_{T_q(\epsilon)}\Delta Y''_{T_q(\epsilon)}\right)$. Using the fact that $\Delta Y'_{T_q(\epsilon)} = 0$, we get the result. □

Lemma A.11 *Under (SH) and (L–s) for* $s<2$, *let*
$\widehat{J}_t'^n = \sum_{i=1}^{[t/\Delta_n]} (\Delta_i^n Y)^2 \frac{\mu_1^{-2}}{(K_n-2)\Delta_n} \sum_{j\in I(n,i)} |\delta_j^n||\delta_{j+1}^n|$. *So clearly,* $\widehat{J}_t'^n = \widehat{J}_t'^{(1)n} - 2\widehat{J}_t'^{(3)n} + \widehat{J}_t'^{(2)n}$. *Then* $\widehat{J}_t^n - \widehat{J}_t'^n \xrightarrow{\mathbb{P}} 0$, *as* $n\to\infty$.

Proof of Lemma A.11 We get

$$\widehat{J}_t^n - \widehat{J}_t'^n = \frac{\mu_1^{-2}}{(K_n-2)\Delta_n} \sum_{i=1}^{[t/\Delta_n]} \sum_{j\in I(n,i)} (\Delta_i^n Y)^2 \left(|\Delta_j^n Y||\Delta_{j+1}^n Y| - |\delta_j^n||\delta_{j+1}^n|\right).$$

From (30) and (31), we deduce that

$$\begin{aligned}
\mathbb{E}_{i-1}^n (\Delta_i^n Y)^2 &= \Delta_n \mathbb{E}_{i-1}^n (\beta_i^n + \chi_i^n)^2 \le 2\Delta_n \mathbb{E}_{i-1}^n \left((\beta_i^n)^2 + \left(\chi_j^n\right)^2\right) \\
&\le K\Delta_n \mathbb{E}_{i-1}^n \left((\beta_i^n)^2 + \left(\chi_j'^n\right)^2 + \left(\chi_j''^n\right)^2\right) \\
&\le \begin{cases} \Delta_n(K+K\Delta_n), & \text{under (L–}s\text{) for } 0\le s\le 1 \\ \Delta_n(K+K\Delta_n+K\Delta_n^{2/\alpha-1}), & \text{under (L–}s\text{) for } 1<s<\alpha<2 \end{cases}.
\end{aligned}$$

Further ,

$$|\Delta_j^n Y||\Delta_{j+1}^n Y| - |\delta_j^n||\delta_{j+1}^n| = |\Delta_j^n Y| \left(|\Delta_{j+1}^n Y| - |\delta_{j+1}^n|\right) + |\delta_{j+1}^n| \left(|\Delta_j^n Y| - |\delta_j^n|\right).$$

So clearly,

$$\begin{aligned}\mathbb{E}_{j-1}^n|\Delta_j^n Y| &\leq \sqrt{\Delta_n}(|\beta_i^n| + |\chi_i^{'n}| + |\chi_i^{''n}|) \\ &\leq \begin{cases} \sqrt{\Delta_n}(K + K\sqrt{\Delta_n}), & \text{under (L–}s\text{) for } 0 \leq s \leq 1 \\ \sqrt{\Delta_n}(K + K\sqrt{\Delta_n} + K\Delta_n^{1/\alpha-1/2}), & \text{under (L–}s\text{) for } 1 < s < \alpha < 2 \end{cases},\end{aligned}$$

and $\mathbb{E}_{j-1}^n|\delta_j^n| \leq K\sqrt{\Delta_n}$ and

$$\begin{aligned}\mathbb{E}_{j-1}^n||\Delta_j^n Y| - |\delta_j^n|| \leq \mathbb{E}_{j-1}^n|\Delta_j^n Y - \delta_j^n| = \sqrt{\Delta_n}\mathbb{E}_{j-1}^n|\chi_i^n| &\leq \sqrt{\Delta_n}\left(\mathbb{E}_{j-1}^n|\chi_i^{'n}| + \mathbb{E}_{j-1}^n|\chi_i^{''n}|\right) \\ &\leq \begin{cases} K\Delta_n & \text{under (L–}s\text{) for } 0 \leq s \leq 1 \\ K(\Delta_n + \Delta_n^{1/\alpha}) & \text{under (L–}s\text{) for } 1 < s < \alpha < 2 \end{cases}.\end{aligned}$$

So we deduce that

$$\mathbb{E}|\widehat{J}_t^n - \widehat{J}_t^{'n}| \leq \frac{K}{(K_n-2)\Delta_n}\frac{t}{\Delta_n}2(K_n-2)\Delta_n^2\sqrt{\Delta_n} = Kt\sqrt{\Delta_n} \to 0, \quad \text{as } n \to \infty,$$

if (L–s) is holds for $0 \leq s \leq 1$ and

$$\begin{aligned}\mathbb{E}|\widehat{J}_t^n - \widehat{J}_t^{'n}| &\leq \frac{K}{(K_n-2)\Delta_n}\frac{t}{\Delta_n}2(K_n-2)\Delta_n\sqrt{\Delta_n}(\Delta_n + \Delta_n^{1/\alpha}) \\ &= Kt(\sqrt{\Delta_n} + \Delta_n^{1/\alpha-1/2}) \to 0, \quad \text{as } n \to \infty,\end{aligned}$$

if (L–s) is holds for $1 < s < \alpha < 2$, which implies the result. □

Lemma A.12 *Theorem 4.6 holds under (SH).*

Proof of Lemma A.12 This result is a direct consequence of the preceding lemmas. □

Proof of Theorem 4.6 The localisation procedure is a standard tool in proving limit theorems for stochastic processes which are observed at discrete time. The main idea is that one can replace local boundedness assumptions as given in (H) by much stronger assumptions (SH), where we assume global boundedness by a (non–random) constant. The proof that our result, which we have proven to hold under (SH) also holds under (H) is omitted, since it goes along the lines of the corresponding proofs in Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006), Jacod (2007*b*, Section 6.3), Aït-Sahalia & Jacod (2006). □

B Tables

		Linear statistic			Ratio statistic			Log–linear statistic		
$[1/\Delta_n]$ (K)	I	Mean	S.D.	Coverage (95%)	Mean	S.D.	Coverage (95%)	Mean	S.D.	Coverage (95%)
39 (35)	3	-0.02	0.84	0.985	0.08	0.94	0.964	-0.03	0.87	0.979
	4	-0.01	0.88	0.979	0.1	1	0.955	-0.02	0.91	0.973
	10	0.03	1	0.964	0.17	1.16	0.932	0.01	1.06	0.941
	20	0.05	1.13	0.926	0.19	1.28	0.893	0.07	1.21	0.889
390 (223)	3	-0.01	0.91	0.976	0	0.92	0.975	-0.03	0.92	0.971
	4	0	0.92	0.972	0	0.93	0.969	-0.02	0.93	0.97
	10	0.01	0.94	0.966	0.02	0.95	0.962	0	0.95	0.962
	20	0.02	0.97	0.96	0.03	0.98	0.955	0	0.97	0.959
1560 (315)	3	0	0.94	0.964	0	0.95	0.963	-0.01	0.95	0.96
	4	0	0.95	0.96	0	0.96	0.958	-0.01	0.96	0.96
	10	0.01	0.98	0.956	0.01	0.98	0.955	0	0.98	0.954
	20	0.02	0.99	0.954	0.02	0.99	0.953	0.01	0.99	0.954
4680 (414)	3	0.01	0.95	0.958	0.01	0.95	0.958	0	0.95	0.959
	4	0.01	0.96	0.961	0.01	0.96	0.96	0	0.96	0.96
	10	0.01	0.97	0.957	0.01	0.97	0.957	0	0.97	0.956
	20	0.01	0.97	0.954	0.01	0.97	0.954	0.01	0.97	0.955

Table 3: Simulation results for BSM: We simulate 5000 replications of $dY_t = \sigma_t dW_t$, where $\sigma_t = \exp(0.125 v_t)$, $dv_t = -0.1 v_t dt + dB_t$, $Cor(W_1, B_1) = -0.62$ over $[0, 1]$. The average value of integrated variance in this simulation study is 1.01. The mean, standard deviation and the 95% coverage of the linear, ratio and log–linear statistic are reported for various numbers of intra–day observations $[1/\Delta_n]$ and various powers I. K denotes the window size of the spot variance estimator.

$[1/\Delta_n]$ (K)	I	Linear statistic Mean	Linear statistic S.D.	Linear statistic Coverage (95%)	Ratio statistic Mean	Ratio statistic S.D.	Ratio statistic Coverage (95%)	Log–linear statistic Mean	Log–linear statistic S.D.	Log–linear statistic Coverage (95%)
		$c = 0.1$								
39 (35)	3	-0.19	0.9	0.965	-0.09	0.98	0.954	-0.2	0.95	0.954
	4	-0.16	0.93	0.965	-0.04	1.03	0.95	-0.18	0.99	0.947
	10	-0.11	1.02	0.948	0.02	1.17	0.931	-0.13	1.11	0.917
	20	-0.08	1.13	0.916	0.05	1.27	0.887	-0.06	1.23	0.879
390 (223)	3	-0.27	0.95	0.951	-0.26	0.96	0.95	-0.3	0.97	0.943
	4	-0.21	0.94	0.96	-0.2	0.95	0.959	-0.23	0.96	0.951
	10	-0.12	0.95	0.963	-0.1	0.96	0.959	-0.14	0.97	0.954
	20	-0.1	0.97	0.957	-0.08	0.98	0.954	-0.11	0.99	0.95
1560 (315)	3	-0.23	0.98	0.947	-0.23	0.99	0.944	-0.25	0.99	0.944
	4	-0.16	0.97	0.952	-0.16	0.97	0.951	-0.18	0.98	0.949
	10	-0.08	0.97	0.955	-0.08	0.97	0.956	-0.09	0.97	0.953
	20	-0.06	0.98	0.953	-0.05	0.98	0.953	-0.07	0.98	0.951
4680 (414)	3	-0.19	0.99	0.947	-0.19	1	0.946	-0.2	1	0.945
	4	-0.12	0.99	0.949	-0.12	0.99	0.948	-0.13	0.99	0.949
	10	-0.04	0.98	0.952	-0.04	0.99	0.952	-0.05	0.99	0.953
	20	-0.03	0.99	0.954	-0.03	0.99	0.952	-0.04	0.99	0.952
		$c = 1$								
39 (35)	3	-0.53	1.08	0.901	-0.53	1.18	0.872	-0.68	1.21	0.848
	4	-0.43	1.06	0.912	-0.37	1.16	0.906	-0.54	1.18	0.87
	10	-0.29	1.1	0.915	-0.13	1.25	0.904	-0.34	1.24	0.871
	20	-0.24	1.17	0.898	-0.08	1.33	0.875	-0.24	1.34	0.852
390 (223)	3	-0.43	1.02	0.923	-0.54	1.06	0.901	-0.58	1.07	0.894
	4	-0.31	1	0.934	-0.35	1.02	0.929	-0.4	1.03	0.921
	10	-0.17	0.99	0.948	-0.15	1	0.947	-0.2	1.01	0.941
	20	-0.13	1	0.948	-0.11	1.01	0.943	-0.15	1.02	0.942
1560 (315)	3	-0.38	1	0.928	-0.49	1.05	0.903	-0.51	1.06	0.9
	4	-0.26	0.98	0.945	-0.3	1	0.937	-0.33	1.01	0.933
	10	-0.13	0.98	0.952	-0.12	0.99	0.95	-0.14	0.99	0.947
	20	-0.1	0.97	0.954	-0.08	0.99	0.953	-0.1	0.99	0.95
4680 (414)	3	-0.33	1.02	0.93	-0.43	1.06	0.915	-0.44	1.06	0.911
	4	-0.21	1.01	0.942	-0.24	1.02	0.94	-0.26	1.02	0.939
	10	-0.09	1.01	0.946	-0.08	1.01	0.942	-0.09	1.01	0.941
	20	-0.07	1.01	0.949	-0.06	1.01	0.944	-0.07	1.01	0.943

Table 4: Simulation results for BSMSCP1: We simulate 5000 replications of $dY_t = \sigma_t dW_t + dL_t$, where $\sigma_t = \exp(0.125 v_t)$, $dv_t = -0.1 v_t dt + dB_t$, $Cor(W_1, B_1) = -0.62$ over $[0, 1]$. L_t is a stratified Poisson process with one jump per day. The jump size is drawn from $\epsilon\sqrt{S}$, where $\epsilon \perp\!\!\!\perp S$, $\epsilon \sim N(0, 1)$ and $S \sim IG(c, c^2/0.25)$. The average value of integrated variance in this simulation study is 1.01. The mean, standard deviation and the 95% coverage of the linear, ratio and log–linear statistic are reported for various numbers of intra–day observations $[1/\Delta_n]$ and various powers I. K denotes the window size of the spot variance estimator.

		Linear statistic			Ratio statistic			Log–linear statistic		
$[1/\Delta_n]$ (K)	I	Mean	S.D.	Coverage (95%)	Mean	S.D.	Coverage (95%)	Mean	S.D.	Coverage (95%)
39	3	-0.38	0.89	0.956	-0.27	0.93	0.962	-0.4	0.96	0.933
(35)	4	-0.34	0.91	0.958	-0.22	0.99	0.958	-0.36	0.99	0.929
	10	-0.26	1	0.942	-0.12	1.12	0.94	-0.28	1.11	0.905
	20	-0.22	1.12	0.907	-0.09	1.23	0.889	-0.21	1.24	0.871
390	3	-0.69	1.02	0.892	-0.68	0.99	0.896	-0.73	1.04	0.875
(223)	4	-0.57	1	0.918	-0.55	0.98	0.921	-0.6	1.02	0.9
	10	-0.4	1	0.938	-0.38	0.99	0.937	-0.43	1.03	0.921
	20	-0.35	1.02	0.934	-0.33	1.01	0.931	-0.38	1.05	0.921
1560	3	-0.78	1.15	0.865	-0.79	1.06	0.865	-0.81	1.11	0.859
(315)	4	-0.61	1.1	0.901	-0.61	1.03	0.902	-0.64	1.07	0.894
	10	-0.4	1.05	0.934	-0.4	1	0.933	-0.42	1.03	0.927
	20	-0.34	1.04	0.938	-0.34	1	0.938	-0.36	1.03	0.933
4680	3	-0.77	1.03	0.869	-0.78	1.03	0.868	-0.79	1.04	0.861
(414)	4	-0.58	1.01	0.903	-0.58	1.01	0.903	-0.6	1.02	0.898
	10	-0.35	0.99	0.934	-0.35	0.99	0.935	-0.36	1	0.932
	20	-0.29	0.99	0.938	-0.29	0.99	0.938	-0.3	1	0.934
7800	3	-0.3	1.01	0.932	-0.31	1.01	0.934	-0.32	1.01	0.932
(470)	4	-0.12	1	0.946	-0.12	1	0.948	-0.13	1	0.948
	10	0.05	0.99	0.952	0.06	0.99	0.95	0.05	0.99	0.951
	20	0.08	0.99	0.949	0.08	0.99	0.949	0.08	0.99	0.95

Table 5: Simulation results for BSMSCP10: We simulate 5000 replications of $dY_t = \sigma_t dW_t + dL_t$, where $\sigma_t = \exp(0.125 v_t)$, $dv_t = -0.1 v_t dt + dB_t$, $Cor(W_1, B_1) = -0.62$ over $[0,1]$. L_t is a stratified Poisson process with ten jumps per day. The jump sizes are drawn from $\epsilon_i \sqrt{S_i}$ for $i = 1, \ldots, 10$, where $\epsilon \perp\!\!\!\perp S$, $\epsilon \sim N(0,1)$ and $S \sim IG(c/10, c^2/(10 * 0.25))$. We set $c = 0.1$. The average value of integrated variance in this simulation study is 1.01. The mean, standard deviation and the 95% coverage of the linear, ratio and log–linear statistic are reported for various numbers of intra–day observations $[1/\Delta_n]$ and various powers I. K denotes the window size of the spot variance estimator.

		Linear statistic			Ratio statistic			Log–linear statistic		
$[1/\Delta_n]$ (K)	I	Mean	S.D.	Coverage (95%)	Mean	S.D.	Coverage (95%)	Mean	S.D.	Coverage (95%)
39 (35)	3	-0.22	0.9	0.964	-0.12	0.98	0.954	-0.24	0.97	0.95
	4	-0.19	0.93	0.965	-0.07	1.03	0.949	-0.21	1	0.945
	10	-0.13	1.02	0.948	0	1.16	0.934	-0.15	1.11	0.916
	20	-0.1	1.15	0.909	0.02	1.28	0.891	-0.09	1.26	0.87
390 (223)	3	-0.28	0.97	0.945	-0.27	0.98	0.942	-0.31	1	0.934
	4	-0.22	0.96	0.949	-0.21	0.97	0.948	-0.25	0.99	0.94
	10	-0.13	0.96	0.954	-0.12	0.97	0.954	-0.16	0.98	0.949
	20	-0.11	0.98	0.953	-0.1	0.99	0.95	-0.13	1	0.948
1560 (315)	3	-0.27	0.99	0.942	-0.28	1	0.941	-0.3	1.01	0.935
	4	-0.21	0.97	0.95	-0.21	0.98	0.95	-0.23	0.98	0.947
	10	-0.12	0.96	0.956	-0.12	0.97	0.955	-0.14	0.97	0.953
	20	-0.1	0.97	0.955	-0.1	0.98	0.953	-0.11	0.98	0.952
4680 (414)	3	-0.26	0.99	0.946	-0.27	1	0.943	-0.28	1	0.94
	4	-0.18	0.97	0.952	-0.18	0.98	0.949	-0.2	0.98	0.949
	10	-0.1	0.97	0.96	-0.1	0.97	0.957	-0.11	0.98	0.956
	20	-0.08	0.97	0.954	-0.08	0.98	0.953	-0.09	0.98	0.953

Table 6: Simulation results for BSMVG: We simulate 5000 replications of $dY_t = \sigma_t dW_t + dL_t$, where $\sigma_t = \exp(0.125 v_t)$, $dv_t = -0.1 v_t dt + dB_t$, $Cor(W_1, B_1) = -0.62$ over $[0,1]$. $L_t \overset{L}{=} \epsilon\sqrt{S}$ is a Variance Gamma process, with $\epsilon \perp\!\!\!\perp S$, $\epsilon \sim N(0,1)$ and $S \sim \Gamma(c/0.25, 1/0.25)$. We set $c = 0.1$. The average value of integrated variance in this simulation study is 1.01. The mean, standard deviation and the 95% coverage of the linear, ratio and log–linear statistic are reported for various numbers of intra–day observations $[1/\Delta_n]$ and various powers I. K denotes the window size of the spot variance estimator.

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