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AN IMPOSSIBILITY RESULT FOR SOCIAL WELFARE RELATIONS IN INFINITELY-LIVED SOCIETIES

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Abstract. This paper extends the analysis of liberal principles in social choice recently proposed by Mariotti and Veneziani ([6]) to societies with an infinite number of agents. First, a novel characterisation of the inegalitarian leximax social welfare relation is provided based on the Individual Benefit Principle, which incorporates a liberal, non-interfering view of society. This result is surprising because the **IBP** has no obvious anti-egalitarian content. Second, it is shown that there exists no weakly complete social welfare relation that satisfies simultaneously the standard axioms of Finite Anonymity, Strong Pareto, and Weak Continuity, and a liberal principle of Non-Interference that generalises **IBP**.

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1. Introduction

Liberal principles in philosophy and social choice tend to express some notion of *individual autonomy or freedom.* In a recent contribution, Mariotti and Veneziani ([6]) have proposed a new axiom - called the Harm Principle (**HP**) - suited for Social Welfare Orderings (SWOs), which is meant to capture a liberal view of non-interference. The basic content of **HP** can be illustrated as follows: consider two welfare allocations u and v such that u is socially preferred to v, and two different welfare allocations u' and v' such that agent i is *worse off* at these than at the corresponding starting allocations, the other agents are equally well off, and agent i prefers u' to v'. Whatever the origin of the decrease in agent i's welfare, **HP** requires that society's preference over u' and v' should agree with person i's preferences: having already suffered a welfare loss in both allocations, and given that no other agent is affected, agent i should not be punished in the SWO by changing social preferences against her.

Although **HP** incorporates no egalitarian content, Mariotti and Veneziani ([6]) have shown that, together with the standard axioms of Anonymity and Strong Pareto, it characterises the leximin SWO in societies with a finite number of agents. Lombardi and Veneziani ([5]) have generalised this counterintuitive result by weakening **HP** and, based on the weak **HP**, they have provided novel characterisations of various SWOs related to Rawls's difference principle, including the maximin and the 'recursive maximin' recently proposed by Roemer ([8],[9]). They have also used the weak **HP** to characterise the leximin social welfare *relation* (SWR) as defined by Asheim and Tungodden ([1]) in economies with an infinite number of agents. The latter result is particularly relevant because the analysis of societies with an infinite number of agents is of focal interest, especially in the discussion of intergenerational justice, but impossibility results easily obtain, for there exists no SWO that satisfies Anonymity and Strong Pareto (see [4]).

This paper extends the analysis of liberal, non-interfering views in societies with an infinite number of agents in two main directions. First, the Individual Benefit Principle (**IBP**) - proposed by Mariotti and Veneziani ([7]) in economies with a finite number of agents - is analysed. The **IBP** also incorporates a liberal, noninterfering view of society and it can be taken as the theoretical complement of **HP**, for it requires society not to switch social preferences when agent i's welfare in both allocations u and v *increases.* Although it has no obvious inegalitarian content, we show that a weaker version of **IBP** suitable for SWRs in infinitely-lived societies, together with other standard axioms, provides a novel characterisation of the inegalitarian *leximax* SWR. This result generalises the characterisation of the leximax SWO in finite societies in ([7]).

Second, as noted by ([7]), **HP** and **IBP** can be taken as two parts of a single liberal view and a weaker version of the principle of Non-Interference ([7]) is proposed, which is suitable for SWRs in societies with an infinite number of agents. An interesting impossibility result for liberal approaches is derived, according to which there exists no weakly complete SWR that satisfies Finite Anonymity, Strong Pareto Optimality, Weak Preference Continuity, and Non-Interference.

2. The framework

Let $X \equiv \mathbb{R}^{\mathbb{N}}$ be the set of countably infinite utility streams, where $\mathbb{R}$ is the set of real numbers and $\mathbb{N}$ is the set of natural numbers. An element of X is ${}_1u = (u_1, u_2, ...)$ and, for $t \in \mathbb{N}$, u_t is the utility level of a representative member of generation t. For $T \in \mathbb{N}$, ${}_1u_T = (u_1, ..., u_T)$ denotes the T-head of ${}_1u$ and ${}_{T+1}u = (u_{T+1}, u_{T+2}, ...)$ denotes the T-tail of ${}_1u$, so that ${}_1u = ({}_1u_T, {}_{T+1}u)$. We write ${}_{con}\epsilon$ for the stream of constant level of well-being equal to $\epsilon \in \mathbb{R}$. A permutation π is a bijective mapping of $\mathbb{N}$ on itself. A permutation π of $\mathbb{N}$ is finite whenever there is $T \in \mathbb{N}$ such that $\pi(t) = t$ for all $t > T$. For any ${}_1u \in X$ and any permutation π, let $\pi({}_1u) = \left(u_{\pi(t)}\right)_{t\in\mathbb{N}}$ be a permutation of ${}_1u$. For any $T \in \mathbb{N}$ and ${}_1u \in X$, ${}_1\bar{u}_T$ is a permutation of ${}_1u_T$ such that the components are ranked in ascending order (i.e., $\bar{u}_1 \leqslant \bar{u}_2 \leqslant ... \leqslant \bar{u}_T$).

For any two utility paths ${}_1u, {}_1v$, we write ${}_1u \geq {}_1v$ to mean $u_t \geq v_t$ for all $t \in \mathbb{N}$; ${}_1u > {}_1v$ to mean ${}_1u \geq {}_1v$ and ${}_1u \neq {}_1v$; and ${}_1u \gg {}_1v$ to mean $u_t > v_t$ for all $t \in \mathbb{N}$.

Let $\succcurlyeq$ be a (binary) relation over X. For any ${}_1u, {}_1v \in X$, we write ${}_1u \succcurlyeq {}_1v$ for $({}_1u, {}_1v) \in \succcurlyeq$ and ${}_1u \not\succcurlyeq {}_1v$ for $({}_1u, {}_1v) \notin \succcurlyeq$; $\succcurlyeq$ stands for "at least as good as". For any ${}_1u, {}_1v \in X$, the asymmetric factor $\succ$ of $\succcurlyeq$ is defined by ${}_1u \succ {}_1v$ if and only if ${}_1u \succcurlyeq {}_1v$ and ${}_1v \not\succcurlyeq {}_1u$, and the symmetric part $\sim$ of $\succcurlyeq$ is defined by ${}_1u \sim {}_1v$ if and only if ${}_1u \succcurlyeq {}_1v$ and ${}_1v \succcurlyeq {}_1u$. They stand, respectively, for "strictly better than" and "indifferent to". A relation $\succcurlyeq$ on X is said to be: *reflexive* if, for any ${}_1u \in X$, ${}_1u \succcurlyeq {}_1u$; *complete* if, for any ${}_1u$, ${}_1v \in X$, ${}_1u \neq {}_1v$ implies ${}_1u \succcurlyeq {}_1v$ or ${}_1v \succcurlyeq {}_1u$; *transitive* if, for any ${}_1u,_1v,_1w \in X$, ${}_1u \succcurlyeq {}_1v \succcurlyeq {}_1w$ implies ${}_1u \succcurlyeq {}_1w$. $\succcurlyeq$ is a quasi-ordering if it is reflexive and transitive, while $\succcurlyeq$ is an ordering if it is a complete quasi-ordering. Let $\succcurlyeq$ and $\succcurlyeq'$ be relations on X. $\succcurlyeq'$ is an extension of $\succcurlyeq$ if $\succcurlyeq \subseteq \succcurlyeq'$ and $\succ \subseteq \succ'$.

3. The Harm Principle and the Leximin SWR

The standard definition of the leximin SWR used in the literature to compare (countably) infinite utility streams is due to Asheim and Tungodden ([1]).

Definition 3.1. (Definition 2, [1], p. 224) For all ${}_1u,_1v \in X$, ${}_1u \sim^{LM} {}_1v \Leftrightarrow \exists \tilde{T} \geq 1$ such that $\forall T \geq \tilde{T}$: ${}_1\bar{u}_T = {}_1\bar{v}_T$, and ${}_1u \succ^{LM} {}_1v \Leftrightarrow \exists \tilde{T} \geq 1$ such that, $\forall T \geq \tilde{T}$, $\exists t \in \{1, ..., T\}$ with $\bar{u}_s = \bar{v}_s$ $(\forall 1 \leq s < t)$ and $\bar{u}_t > \bar{v}_t$.

The characterisation of the leximin derived by ([5]) focuses on definition 3.1, and it is based on the following axioms.[1]

Finite Anonymity, **FA**: $\forall {}_1u \in X$ and $\forall$ finite permutation π of $\mathbb{N}$, $\pi({}_1u) \sim {}_1u$.

Strong Pareto Optimality, **SPO**: $\forall {}_1u,_1v \in X : {}_1u > {}_1v \Rightarrow {}_1u \succ {}_1v$.

Weak Preference Continuity, **WPC**: $\forall {}_1u,_1v \in X :$ $\exists \tilde{T} \geq 1$ such that $({}_1u_T,_{T+1}v) \succ {}_1v$ $\forall T \geq \tilde{T} \Rightarrow {}_1u \succ {}_1v$.

Weak Completeness, **WC**: $\forall {}_1u$, ${}_1v \in X$, $\exists T \geq 1$ $\pi({}_1u_T,_{T+1}v) \neq {}_1v$ $\forall$ finite permutation π of $\mathbb{N} \Rightarrow ({}_1u_T,_{T+1}v) \succcurlyeq {}_1v$ or ${}_1v \succcurlyeq ({}_1u_T,_{T+1}v)$.

Harm Principle, **HP**: $\forall {}_1u,_1v,_1u',_1v' \in X : \exists T \geq 1$ ${}_1u = ({}_1u_T,_{T+1}v) \succ {}_1v$, and ${}_1u'$ and ${}_1v'$ are such that, $\exists i \leq T$,

[1] Definition 3.1 is also known as the W-Leximin ([1], p.224). [5] also provide a characterisation of the S-Leximin ([1], p.224) and of the leximin SWR as defined by ([2]). Analogous impossibility results can be proved for the latter definitions.

$$\begin{aligned} u'_i &< u_i \\ v'_i &< v_i \\ u'_j &= u_j \ \forall j \neq i \\ v'_j &= v_j \ \forall j \neq i \end{aligned}$$

implies ${}_1u' \succcurlyeq {}_1v'$ whenever $u'_i > v'_i$.

FA and **SPO** are standard and need no further comment. **WPC** has been proposed by Asheim and Tungodden ([1], p. 223) and it represents a mainly technical, weak requirement to deal with infinite-dimensional vectors. **WC** states that a SWR should be able to compare vectors with the same tail: this seems an obviously desirable property, as it imposes a minimum requirement of completeness. Finally, **HP** formalises the Harm Principle in societies with an infinite number of agents. It is weaker than the version proposed by ([7]), because it does not require that ${}_1u' \succ {}_1v'$ and moreover it only holds for vectors with the same tail.[2] Lombardi and Veneziani ([5]) have proved the following Theorem.

Theorem 3.2. *(Theorem 3.5,* [5]*, p. 12)* $\succcurlyeq$ *is an extension of* $\succcurlyeq^{LM}$ *if and only if* $\succsim$ *satisfies* ***FA****,* ***SPO****,* ***HP****,* ***WPC****, and* ***WC****.*

As noted in [6] and [5], a characterisation of the leximin based on **HP** is surprising, because **HP** has no obvious egalitarian content, unlike the standard axiom of Hammond Equity (see, e.g., [3], and [1]). It is also quite surprising that, by a suitable change in the axiom incorporating a liberal view of non-interference, it is possible to characterise the strongly *inegalitarian* leximax SWR.

4. The Benefit Principle and the Leximax SWR

According to the leximax, that society is best which (lexicographically) maximises the welfare of its best-off members. In economies with an infinite number of agents, this intuition can be formalised as follows.

Definition 4.1. For all ${}_1u, {}_1v \in X$, ${}_1u \sim^{LX} {}_1v \Leftrightarrow \exists \tilde{T} \geq 1$ such that $\forall T \geq \tilde{T}$: ${}_1\bar{u}_T = {}_1\bar{v}_T$, and ${}_1u \succ^{LX} {}_1v \Leftrightarrow \exists \tilde{T} \geq 1$ such that, $\forall T \geq \tilde{T}$, $\exists t \in \{1, ..., T\}$ with $\bar{u}_s = \bar{v}_s$ $(\forall t < s \leq T)$ and $\bar{u}_t > \bar{v}_t$.

[2] For a detailed discussion of the axioms, see ([5]).

In order to characterise the leximax SWR, the same axioms as for the leximin are used, except for **HP**, which is substituted with the Individual Benefit Principle. The **IBP** also captures a liberal requirement of noninterference and can be formalised as follows.

INDIVIDUAL BENEFIT PRINCIPLE, **IBP**:$\forall {}_1u, {}_1v, {}_1u', {}_1v' \in X : \exists T \geq 1$ ${}_1u = ({}_1u_T, {}_{T+1}v) \succ {}_1v$, and ${}_1u'$ and ${}_1v'$ are such that, $\exists i \leq T$,

$$\begin{aligned} u_i' &> u_i \\ v_i' &> v_i \\ u_j' &= u_j \ \forall j \neq i \\ v_j' &= v_j \ \forall j \neq i \end{aligned}$$

implies ${}_1u' \succcurlyeq {}_1v'$ whenever $u_i' > v_i'$.

In other words, consider two alternatives ${}_1u$ and ${}_1v$, whereby ${}_1u$ is socially preferred to ${}_1v$, and two different welfare allocations ${}_1u'$ and ${}_1v'$ such that agent i is better off at these than at the corresponding starting allocations, the other agents are equally well-off, and i prefers ${}_1u'$ to ${}_1v'$. **IBP** requires that society's preference over ${}_1u'$ and ${}_1v'$ should agree with person i's preferences: although i's welfare has increased in both allocations, society should not 'punish' i by reversing social preferences. The moral intuition behind **IBP** is similar to the **HP**, and yet the next Theorem proves that the **IBP** leads to an extremely different result.

Theorem 4.2. *$\succcurlyeq$ is an extension of $\succcurlyeq^{LX}$ if and only if $\succcurlyeq$ satisfies **FA**, **SPO**, **IBP**, **WPC**, and **WC**.*

Proof. ($\Rightarrow$) Let $\succcurlyeq^{LX} \subseteq \succcurlyeq$. It is easy to see that $\succcurlyeq$ meets **FA**, **SPO**, **WPC**, and **WC**. We show that $\succcurlyeq$ satisfies **IBP**. Take any ${}_1u$, ${}_1v$, ${}_1u'$, ${}_1v' \in X$ such that ${}_1u = ({}_1u_T, {}_{T+1}v) \succ {}_1v$ $\exists T \geq 1$, and ${}_1u'$, ${}_1v'$ are such that, $\exists i \leq T$, $u_i' > u_i$, $v_i' > v_i$, $u_j' = u_j \ \forall j \neq i$, $v_j' = v_j \ \forall j \neq i$. We show that ${}_1u' \succcurlyeq {}_1v'$ whenever $u_i' > v_i'$. As ${}_1u$, ${}_1v$ have the same tail, ${}_1u \succ^{LX} {}_1v$. Then, $\exists \tilde{T} \geq 1$ such that, $\forall T' \geq \tilde{T}$, $\exists t \in \{1, ..., T'\}$ with $\bar{u}_s = \bar{v}_s \ \forall t < s \leq T'$ and $\bar{u}_t > \bar{v}_t$. Consider any $T' \geq \tilde{T}$. If $\bar{u}_{T'} > \bar{v}_{T'}$, the result follows as $\bar{u}'_{T'} \in \{u_i', \bar{u}_{T'}\}$ and $\bar{v}'_{T'} \in \{v_i', \bar{v}_{T'}\}$. Therefore suppose $\bar{u}_{T'} = \bar{v}_{T'}$. If $\bar{v}_\ell = \bar{v}'_\ell$ for all $t \leq \ell \leq T'$, the result follows. Otherwise, let $\bar{v}_\ell \neq \bar{v}'_\ell$ for some $t \leq \ell \leq T'$. We distinguish two cases.

Case 1. $\bar{v}_t < v'_i < \bar{v}_{t+1}$

Then, $\bar{v}'_{t+1} > \bar{v}'_t = v'_i > \bar{v}_t$ and $\bar{v}'_s = \bar{v}_s$ for all $T' \geq s > t$. If $u'_i \in (v'_i, \bar{u}_{t+1}]$, then $\bar{u}'_t > v'_i = \bar{v}'_t$. Otherwise, let $u'_i > \bar{u}_{t+1}$. Thus, there exists $j \geq t+1$ such that $u'_i = \bar{u}'_j > \bar{u}_j$. Let

$$m = \max\left\{t+1 \leq j \leq T' | \bar{u}'_j > \bar{u}_j\right\}.$$

Since $\bar{v}'_j = \bar{v}_j = \bar{u}_j\ \forall t < j \leq T'$ it follows that $\bar{u}'_m > \bar{v}'_m$. In both cases, there exists $t^* \leq T'$ such that $\bar{u}'_s = \bar{v}'_s\ \forall t^* < s \leq T'$ and $\bar{u}'_{t^*} > \bar{v}'_{t^*}$.

Case 2. $v'_i \geq \bar{v}_{t+1}$

If $v'_i \geq \bar{v}_{T'}$, then $\bar{u}'_{T'} > \bar{v}'_{T'}$ as $u'_i > v'_i$. Otherwise, let $v'_i < \bar{v}_{T'}$. Let

$$\ell = \min\left\{t+1 < j \leq T' | v'_i < \bar{v}_j\right\}.$$

Then, $\bar{v}_\ell > v'_i = \bar{v}'_{\ell-1} \geq \bar{u}_{\ell-1} = \bar{v}_{\ell-1}$. As $v'_i < u'_i$, it follows that $u'_i > \bar{u}_{\ell-1}$. If $u'_i \in (\bar{u}_{\ell-1}, \bar{u}_\ell]$, then $\bar{v}'_{\ell-1} < u'_i = \bar{u}'_{\ell-1}$. Otherwise, let $\bar{u}_\ell < u'_i$. Then, there exists $\ell \leq m \leq T'$ such that $\bar{v}'_m = \bar{v}_m < \bar{u}'_m = u'_i$ and if $m < T'$, $\bar{u}'_s = \bar{v}'_s\ \forall m < s \leq T'$. In both cases, there exists $t^* \leq T'$ such that $\bar{u}'_s = \bar{v}'_s\ \forall t^* < s \leq T'$ and $\bar{u}'_{t^*} > \bar{v}'_{t^*}$.

Since it holds for any $T' \geq \tilde{T}$, we have that ${}_1u' \succcurlyeq {}_1v'$ as $\succcurlyeq^{LX} \subseteq \succcurlyeq$.

($\Leftarrow$) Suppose that $\succcurlyeq$ satisfies **FA**, **SPO**, **IBP**, **WPC**, and **WC**. We show that $\sim^{LX^*} \subseteq \sim$ and $\succ^{LX^*} \subseteq \succ$. Take any ${}_1u$, ${}_1v \in X$. If ${}_1u \sim^{LX} {}_1v$, then ${}_{T+1}u = {}_{T+1}v\ \forall T > \tilde{T}$, so **FA** implies ${}_1u \sim {}_1v$.

Next, we show that ${}_1u \succ {}_1v$ whenever ${}_1u \succ^{LX} {}_1v$. Thus, suppose that ${}_1u \succ^{LX} {}_1v$. Take any $T \geq \tilde{T}$ and consider the vector ${}_1w \equiv ({}_1u_T, {}_{T+1}v)$. We want to show that ${}_1w \succ {}_1v$. By **FA** and transitivity, we can consider ${}_1\bar{w} \equiv ({}_1\bar{u}_T, {}_{T+1}v)$ and ${}_1\bar{v} \equiv ({}_1\bar{v}_T, {}_{T+1}v)$. Suppose that ${}_1\bar{v} \succcurlyeq {}_1\bar{w}$. We distinguish two cases.

Case 1. ${}_1\bar{v} \succ {}_1\bar{w}$

By **SPO** it follows that $\bar{v}_l > \bar{w}_l$, some $l < t \leq T$. Let

$$k = \max\{1 \leq l < t | \bar{v}_l > \bar{w}_l\}.$$

By **FA**, let $w_i = \bar{w}_k$ and $v_i = \bar{v}_{k+g}$ for some $0 < g \leq t-k$ with $\bar{w}_{k+g} > \bar{v}_{k+g}$. Let $d_1, d_2 > 0$, and consider vectors ${}_1w'$, ${}_1v'$ formed from ${}_1\bar{w}$, ${}_1\bar{v}$ as follows: $\bar{v}_{k+g}$ is raised to $\bar{v}_{k+g}+d_1$ such that $\bar{w}_{k+g} > \bar{v}_{k+g}+d_1$; $\bar{w}_k$ is raised to $\bar{w}_k+d_2$ such that $\bar{v}_{k+g}+d_1 > \bar{w}_k+d_2 > \bar{v}_k$; and all other entries of ${}_1\bar{w}$ and ${}_1\bar{v}$ are unchanged. By **FA**, consider ${}_1\bar{w}' = ({}_1\bar{w}'_T, {}_{T+1}v)$ and ${}_1\bar{v}' = ({}_1\bar{v}'_T, {}_{T+1}v)$. By construction $\bar{w}'_j \geq \bar{v}'_j$ for all $T \geq j \geq k$, with $\bar{w}'_{k+g} > \bar{v}'_{k+g}$ and $\bar{w}'_k > \bar{v}'_k$. **IBP** implies ${}_1\bar{v}' \succcurlyeq {}_1\bar{w}'$, and by **SPO** d_1, d_2 can be chosen so that ${}_1\bar{v}' \succ {}_1\bar{w}'$, without loss of generality. Consider two cases:

a) Suppose that $\bar{v}_k > \bar{w}_k$, but $\bar{w}_l \geq \bar{v}_l$ for all $l < k$. It follows that ${}_1\bar{w}' > {}_1\bar{v}'$, and so **SPO** implies that ${}_1\bar{w}' \succ {}_1\bar{v}'$, a contradiction.

b) Suppose that $\bar{v}_l > \bar{w}_l$ for some $l < k$. Note that by construction $\bar{v}'_l = \bar{v}_l$ and $\bar{w}'_l = \bar{w}_l$ for all $l < k$. Then, let

$$k' = \max\{1 \leq l < k | \bar{v}'_l > \bar{w}'_l\}.$$

The above argument can be applied to ${}_1\bar{w}'$, ${}_1\bar{v}'$ to derive vectors ${}_1\bar{w}''$, ${}_1\bar{v}''$ such that $\bar{w}''_j \geq \bar{v}''_j$ for all $j \geq k'$, whereas by **IBP** and **SPO** ${}_1\bar{v}'' \succ {}_1\bar{w}''$. And so on. After a finite number of iterations q, two vectors ${}_1\bar{w}^q$, ${}_1\bar{v}^q$ can be derived such that, by **IBP** and **SPO**, ${}_1\bar{v}^q \succ {}_1\bar{w}^q$ but, by **SPO**, ${}_1\bar{w}^q \succ {}_1\bar{v}^q$, yielding the desired contradiction.

Case 2. ${}_1\bar{v} \sim {}_1\bar{w}$

By assumption, $\bar{v}_t < \bar{u}_t \equiv \bar{w}_t$. Therefore, define ${}_1\bar{w}'$ as follows: $\bar{w}'_\tau = \bar{w}_\tau$ $\forall \tau \in \mathbb{N}\backslash\{t\}$ and $\bar{w}'_t = \bar{w}_t - \epsilon > \bar{v}_t$, some $\epsilon > 0$. By **SPO** and transitivity, it follows that ${}_1\bar{v} \succ {}_1\bar{w}'$ but ${}_1\bar{w}' \succ^{LX} {}_1\bar{v}$. Hence, the argument of *Case* 1 above can be applied to ${}_1\bar{v}$ and ${}_1\bar{w}'$, yielding the desired contradiction.

As ${}_1\bar{v} \not\succcurlyeq {}_1\bar{w}$ **WC** implies ${}_1\bar{w} \succ {}_1\bar{v}$. **FA** and transitivity imply that $({}_1u_T, {}_{T+1}v) \succ {}_1v$. Since this is true for any $T \geq \tilde{T}$, **WPC** implies ${}_1u \succ {}_1v$. $\square$

Theorem 4.2 has an interesting theoretical implication. Consider the following axiom of Non-Interference, which incorporates the normative intuitions behind **HP** and **IBP** in a unified liberal framework, and generalises the principle of Non-Interference proposed by ([7]) to economies with an infinite number of agents.

NON-INTERFERENCE, **NI**:$\forall {}_1u, {}_1v, {}_1u', {}_1v' \in X : \exists T \geq 1$ ${}_1u = ({}_1u_T, {}_{T+1}v) \succ {}_1v$, and ${}_1u'$ and ${}_1v'$ are such that, $\exists i \leq T$,

$$\begin{aligned} (u'_i - u_i)(v'_i - v_i) &> 0 \\ u'_j &= u_j \; \forall j \neq i \\ v'_j &= v_j \; \forall j \neq i \end{aligned}$$

implies ${}_1u' \succcurlyeq {}_1v'$ whenever $u'_i > v'_i$.

As is well-known, there exists no SWO defined on an infinite bounded set of real vectors which satisfies Anonymity and **SPO** (see, [4]). Theorems 3.2 and 4.2 imply that there is no weakly complete SWR that satisfies **FA**, **SPO**, **WPC**, and **NI**.

Theorem 4.3. *There exists no* SWR *on X that satisfies **FA**, **SPO**, **WPC**, **WC** and **NI**.*

Proof. By contradiction. Let $\nu, \mu \in \mathbb{R}$ with $\nu > \mu$ and consider vectors ${}_1u, {}_1v \in X$ such that ${}_1u = {}_{con}\mu$ and ${}_1v = (v_1, {}_2v)$, where $v_1 < \mu$ and ${}_2v = {}_{con}\nu$. By Theorem 3.2, ${}_1u \succ^{LM} {}_1v$, so that ${}_1u \succ {}_1v$, but by Theorem 4.2, ${}_1v \succ^{LX} {}_1u$, so that ${}_1v \succ {}_1u$, a contradiction. □

Conclusions

This paper analyses liberal axioms for SWRs in societies with an infinite number of agents. The leximax SWR is characterised by appealing to the Individual Benefit Principle, which incorporates a liberal, non-interfering view of society. This result is interesting per se, since it provides the first characterisation of the leximax in economies with an infinite number of agents, and because the **IBP** has no obvious anti-egalitarian content. It also has relevant implications for liberal approaches to social choice. For it allows us to show that there exists no weakly complete SWR that satisfies the standard axioms of Finite Anonymity, Strong Pareto, a weak requirement on continuity, and the liberal principle of Non-Interference.

References

[1] Asheim, G. B. and B. Tungodden (2004) "Resolving distributional conflicts between generations", *Economic Theory* 24: 221-230.

[2] Bossert, W., Sprumont, Y., and K. Suzumura (2007) "Ordering infinite utility streams", *Journal of Economic Theory* 135: 179-189.

[3] Hammond, P. (1976) "Equity, Arrow's Conditions and Rawls' Difference Principle", *Econometrica* 44: 793-804.

[4] Lauwers, L. (1997) "Rawlsian equity and generalised utilitarianism with an infinite population", *Economic Theory* 9: 143-150.

[5] Lombardi, M. and R. Veneziani (2009) "Liberal Egalitarianism and the Harm Principle", Mimeo, Queen Mary University of London.

[6] Mariotti, M. and R. Veneziani (2009) "Non-Interference Implies Equality", *Social Choice and Welfare*, January, 32(1): 123-128.

[7] Mariotti, M. and R. Veneziani (2009b) "The Impossibility of Non-Interference", Mimeo, Queen Mary University of London.

[8] Roemer, J.E. (2008) "Notes on Intergenerational Justice and Sustainability" in J.E. Roemer and K. Suzumura (eds.) *Intergenerational Equity and Sustainability*, MacMillan, London.

[9] Roemer, J.E. and R.Veneziani (2008) "Intergenerational Justice, International Relations, and Sustainability" in J.E. Roemer and K.Suzumura (eds.) *Intergenerational Equity and Sustainability*, MacMillan, London, pp. 228-251.

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1. Appendix: Independence of axioms

Let Π be the set of all finite permutations.

For an example violating only **FA**, define $\succcurlyeq$ on X in the following way: $\forall {}_1x,\ {}_1y \in X$

$$1)\ {}_1x = {}_1y \Rightarrow {}_1x \sim {}_1y$$
$$2)\ {}_1x \neq {}_1y \text{ and } {}_1x = \pi({}_1y)\, \exists\pi \in \Pi : {}_1u \not\succcurlyeq {}_1v \text{ and } {}_1v \not\succcurlyeq {}_1u$$
$$3)\ {}_1x \neq {}_1y \text{ and } {}_1x \neq \pi({}_1y)\, \forall\pi \in \Pi\text{: } {}_1x \succcurlyeq^{LX^*} {}_1y \Rightarrow {}_1x \succcurlyeq {}_1y.$$

The SWR $\succcurlyeq$ on X is not an extension of the leximin SWR $\succcurlyeq^{LX^*}$. The SWR $\succcurlyeq$ on X satisfies all properties except **FA**.

For an example violating only **SPO**, for all ${}_1x,\ {}_1y \in X$, define $\succcurlyeq$ on X in the following way: ${}_1x \sim {}_1y$. The SWR $\succcurlyeq$ on X is not an extension of the leximax SWR $\succcurlyeq^{LX^*}$. Clearly, the SWR $\succcurlyeq$ on X satisfies all properties except **SPO**.

For an example violating only **WC**, for all ${}_1x, {}_1y \in X$, define $\succcurlyeq$ on X in the following way: ${}_1x \succ {}_1y$ if ${}_1x > \pi({}_1y)\, \exists\pi \in \Pi$; ${}_1x \sim {}_1y$ if ${}_1x = \pi({}_1y)\, \exists\pi \in \Pi$; and ${}_1x \not\succcurlyeq {}_1y$ and ${}_1y \not\succcurlyeq {}_1x$ if ${}_1x \not> \pi({}_1y)$, ${}_1y \not> \pi({}_1x)$, and ${}_1x \neq \pi({}_1y)\ \forall\pi \in \Pi$. The SWR $\succcurlyeq$ on X is not an extension of the leximax SWR $\succcurlyeq^{LX^*}$. Clearly, the SWR $\succcurlyeq$ on X satisfies all properties except **WC**.

For an example violating only **WPC**, define $\succcurlyeq$ on X in the following way: $\forall {}_1x,\ {}_1y \in X$

$$\nexists T \geq 1 \text{ s.t. } {}_Tx = {}_Ty,\ {}_1x \not> {}_1y \text{ and } {}_1y \not> {}_1x \Rightarrow {}_1x \not\succcurlyeq {}_1y \text{ and } {}_1y \not\succcurlyeq {}_1x,$$

otherwise,

$${}_1x \succcurlyeq^{LX^*} {}_1y \Rightarrow {}_1x \succcurlyeq {}_1y.$$

$\succcurlyeq$ on X is a SWR. Fix $\mu, \nu, \epsilon, \delta \in \mathbb{R}$, with $\mu > \delta > \epsilon > \nu$. Let ${}_1x = (\delta,\ {}_{con}\epsilon)$ and ${}_1y = (\mu,\ {}_{con}\nu)$. Clearly, ${}_1x,\ {}_1y \in X$, and $({}_1y_T,\ {}_{T+1}x) \succ^{LX^*} {}_1x\ \forall T \geq 2$, ${}_1y \succ^{LX^*} {}_1x$, but ${}_1x \not\succcurlyeq {}_1y$ and ${}_1y \not\succcurlyeq {}_1x$. It follows that the SWR $\succcurlyeq$ on X is not an extension of the leximax SWR. The SWR $\succcurlyeq$ on X satisfies all properties except **WPC**.

For an example violating only **IBP***, define $\succcurlyeq$ on X in the following way: $\forall {}_1x, {}_1y \in X$

$${}_1x \sim {}_1y \Leftrightarrow \exists\tilde{T} \geq 1 \text{ s.t. } \forall T \geq \tilde{T} : {}_1\bar{x}_T = {}_1\bar{y}_T,$$

and

$${}_1x \succ {}_1y \Leftrightarrow \exists\tilde{T} \geq 1 \text{ s.t. } \forall T \geq \tilde{T} : \exists t \in \{1, ..., T\}\ \bar{u}_s = \bar{v}_s\ (\forall 1 \leq s < t) \text{ and } \bar{u}_t > \bar{v}_t.$$

$\succcurlyeq$ on X is a SWR (i.e., the w-leximin SWR). It follows that the SWR $\succcurlyeq$ on X is not an extension of the leximax SWR. The SWR $\succcurlyeq$ on X satisfies all properties except **IBP***.

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