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ROBUST EXPONENTIAL HEDGING IN A BROWNIAN SETTING

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This paper studies the robust exponential hedging in a Brownian factor model, giving a solvable example using a PDE argument. The dual problem is reduced to a standard stochastic control problem, of which the HJB equation admits a classical solution. Then an optimal strategy will be expressed in terms of the solution to the HJB equation.

1. INTRODUCTION

This short article aims to provide a solvable example for the robust exponential hedging problem studied by Owari [5]:

$$\text{(1.1)} \qquad \text{minimize} \quad \sup_{P\in\mathcal{P}} E^P[e^{-\alpha(\theta\cdot S_T-H)}], \quad \text{among } \theta\in\Theta.$$

Here S is a d-dimensional càdlàg locally bounded semimartingale on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},R)$, $\mathcal{P}$ is a convex set of probability measures absolutely continuous w.r.t. R, H is a random variable, and Θ is a set of d-dimensional predictable (S,R)-integrable processes. The set $\mathcal{P}$ is a mathematical expression of *model uncertainty*, and the problem is equivalent to maximize the *robust exponential utility* from the net terminal wealth for the seller of the claim H.

The problem (1.1) is solved via its dual:

$$\text{(1.2)} \qquad \text{minimize} \quad \mathcal{H}(Q|P)-\alpha E^Q[H], \quad \text{among } (Q,P)\in\mathcal{Q}_f\times\mathcal{P},$$

where $\mathcal{H}(\cdot|\cdot)$ denotes the relative entropy, and $\mathcal{Q}_f$ is the set of R-absolutely continuous local martingale measures for S, which have finite relative entropy with some $P\in\mathcal{P}$.

Assume:

(A1) $\{dP/dR : P\in\mathcal{P}\}$ is weakly compact in $L^1(R)$.
(A2) $\mathcal{Q}_f^e(S) := \{Q\in\mathcal{Q}_f : Q\sim R\}\neq\emptyset$.
(A3) $\{e^{\alpha|H|}dP/dR : P\in\mathcal{P}\}$ is uniformly integrable and

$$\sup_{P\in\mathcal{P}} E^P[e^{(\alpha+\varepsilon)|H|}]<\infty, \quad \exists\varepsilon>0.$$

Under (A1)–(A3), [5] shows that (1.2) admits a maximal solution, i.e., there exists a pair $(\widehat{Q}_H,\widehat{P}_H)\in\mathcal{Q}_f\times\mathcal{P}$ which satisfies

$$\text{(1.3)} \qquad \mathcal{H}(\widehat{Q}_H|\widehat{P}_H)-\alpha E^{\widehat{Q}_H}[H]=\inf_{(Q,P)\in\mathcal{Q}_f\times\mathcal{P}}(\mathcal{H}(Q|P)-\alpha E^Q[H]),$$

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and if $(\widetilde{Q}, \widetilde{P}) \in \mathcal{Q}_f \times \mathcal{P}$ also satisfies (1.3), then $\widetilde{P} \ll \widehat{P}_H$ and $d\widetilde{Q}/d\widetilde{P} = d\widehat{Q}_H/d\widehat{P}_H$, $\widetilde{P}$-a.s. This solution has a kind of martingale representation:

$$\frac{d\widehat{Q}_H}{d\widehat{P}_H} = e^{-\alpha(\hat{\theta}\cdot S_T - H)}/E^{\widehat{P}_H}[e^{-\alpha(\hat{\theta}\cdot S_T - H)}], \quad \widehat{Q}_H\text{-a.s.} \tag{1.4}$$

where $\hat{\theta}$ is a predictable $(S, \widehat{Q}_H)$-integrable process such that $\hat{\theta} \cdot S$ is a $\widehat{Q}_H$-martingale. Finally, if we assume additionally:

(A4) $\widehat{Q}_H \sim R$,

the strategy $\hat{\theta}$ is shown to be optimal for (1.1) with the admissible class:

$$\Theta_H := \{\theta \in L(S) : \ \theta \cdot S \text{ is a martingale under } \forall Q \in \mathcal{Q}_f(\widehat{P}_H)\},$$

where $\mathcal{Q}_f(\widehat{P}_H)$ denotes the set of elements of $\mathcal{Q}_f$ which have a finite relative entropy with $\widehat{P}_H$.

In the sequel, we investigate this problem in a specific setting for which the optimal strategy $\hat{\theta}$ is explicitely represented, using a standard stochastic control technique.

2. Main Results

This section states the main results of this paper. All proofs are collected in Section 4.

2.1. Setup

Let $W = (W^1, W^2)$ be a 2-dimensional R-Brownian motion, and $(\mathcal{F}_t)_{t\in[0,T]}$ its augmented natural filtration. Suppose that the price process S is given by the SDE:

$$\begin{aligned} dS_t &= S_t(b(Y_t)dt + \sigma(Y_t)dW_t^1), \\ dY_t &= g(Y_t)dt + \rho dW_t^1 + \bar{\rho} dW_t^2, \end{aligned} \tag{2.1}$$

where $\rho \in [-1, 1]$ and $\bar{\rho} := \sqrt{1-\rho^2}$. The set $\mathcal{P}$ of candidate models is given as follows. Let C be a convex compact subset of $\mathbb{R}^2$ containing the origin, and $\mathcal{I}_{\mathcal{P}}$ the set of 2-dimensional predictable C-valued processes. Then we set

$$\mathcal{P} := \{P^\nu \sim R : \ dP^\nu/dR = \mathcal{E}_T(-\nu \cdot W), \nu \in \mathcal{I}_{\mathcal{P}}\}, \tag{2.2}$$

where $\mathcal{E}(M) := \exp(M - \langle M\rangle/2)$ denotes the *Doléans-Dade exponential* of a continuous local martingale M. Finally, the claim H is assumed to be of the form $H = h(Y_T)$ for a measurable function h.

Remark 2.1. A typical situation underlying our setup is as follows. A financial institution sells an option written on an untradable index Y, and want to maximize her utility by trading an asset S which is correlated to Y. However, the probabilistic model of assets (S, Y) is uncertain in its expected rate of return (drift, in mathematical language). Actually, the dynamics under the probability P^ν is:

$$\begin{aligned} dS_t &= S_t((b(Y_t) - \nu_t^1\sigma(Y_t))dt + \sigma(Y_t)dW_t^{1,\nu}) \\ dY_t &= (g(Y_t) - \rho\nu_t^1 - \bar{\rho}\nu_2)dt + \rho dW_t^{1,\nu} + \bar{\rho} dW_t^{2,\nu}. \end{aligned}$$

In this context, we can know only the *range* of the drift through the set C appearing in the definition of $\mathcal{P}$.

In what follows, we assume

(B1) $b, \sigma, g \in C_b^2(\mathbb{R}) := \{f \in C^2(\mathbb{R}) : \ f, f', f'' \text{ are bounded}\}$.
(B2) There exists a constant $k > 0$ such that $\sigma(y) \geq k$ for all $y \in \mathbb{R}$.

(B3) $h \in C^2(\mathbb{R})$, h' is bounded and h'' has a polynomial growth.

Our first task is to check that:

Lemma 2.2. *Under (B1) – (B3), the conditions (A1) – (A4) of [5] are satisfied.*

Once this lemma is established, an optimal strategy $\hat{\theta}$ will be derived via (1) solving the dual problem (1.2), and (2) finding $\hat{\theta}$ satisfying (1.4).

Remark 2.3.

1. In this setting, we can show (see Appendix A) that

$$\mathcal{H}(Q|P) < \infty \text{ for } \exists P \in \mathcal{P} \Leftrightarrow \mathcal{H}(Q|R) < \infty, \tag{2.3}$$

   for all local martingale measures Q. In particular, Θ_H is characterized as the class of predictable (S, R)-integrable processes θ such that $\theta \cdot S$ is a martingale under all absolutely continuous local martingale measures Q with $\mathcal{H}(Q|R) < \infty$. This condition is further reduced to "all *equivalent* martingale measures with...". Therefore, the class Θ_H is actually independent of $\widehat{P}_H$, hence of H. This point is conceptually important since the dependence of Θ on $\widehat{P}_H$, which is a part of the *solution* to the dual problem, implies that we can not specify the admissible class for the primal problem until we solve the dual problem.
2. Next for our purpose, it suffices to consider $\mathcal{Q}_f^e$ for the domain of dual problem because we already know that a solution to the dual problem is obtained in $\mathcal{Q}_f^e \times \mathcal{P}$, and are interested only in *representing* $\hat{\theta}$. In our setup, this class admits an explicit representation:

$$\begin{aligned} &\mathcal{Q}_f^e = \{Q^\eta : dQ^\eta/dR = \mathcal{E}_T(-(\lambda(Y), \eta) \cdot W), \eta \in \mathcal{I}_M\}, \\ &\mathcal{I}_M := \{\eta : \text{predictable}, E^R[\textstyle\int_0^T \eta_t^2 dt] < \infty, E^R[\mathcal{E}_T(-(\lambda(Y), \eta) \cdot W)] = 1\}. \end{aligned} \tag{2.4}$$

   where $\lambda := b/\sigma$.

2.2. Dual Problem

Let

$$J_t^{\eta,\nu} := E^\eta[\alpha h(Y_T) - \frac{1}{2}\int_t^T \|\nu_s - (\lambda(Y_s), \eta_s)'\|^2 ds|\mathcal{F}_t], \quad t \in [0, T],$$

where $E^\eta[\cdot]$ denotes the expectation under Q^η, " $'$ " is the transpose, and $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^2$. The dual problem (1.2) is now reduced to the following stochastic control problem:

$$(\mathscr{D}) \qquad \text{maximize} \quad J_0^{\eta,\nu} \quad \text{among } (\eta, \nu) \in \mathcal{I}_M \times \mathcal{I}_{\mathcal{P}}.$$

For each constant $\eta \in \mathbb{R}$, set

$$\begin{aligned} \mathcal{A}^\eta &:= (g - \rho\lambda - \bar{\rho}\eta)\partial_y + \frac{1}{2}\partial_{yy} \\ &= \mathcal{A}^0 - \bar{\rho}\eta\partial_y. \end{aligned} \tag{2.5}$$

where $\partial_y := \partial/\partial y$ and $\partial_{yy} := \partial^2/\partial y^2$ etc. Then the HJB equation for $(\mathscr{D})$ is formally given by

$$\begin{cases} v_t + \sup_{(\eta,\nu)\in\mathbb{R}\times C}(\mathcal{A}^\eta v - \frac{1}{2}\|\nu - (\lambda, \eta)'\|^2) = 0 \\ v(T, y) = \alpha h(y), \end{cases} \tag{2.6}$$

Theorem 2.4. *The HJB equation (2.6) admits a unique classical solution $v \in C^{1,2}((0,T)\times\mathbb{R}) \cap C([0,T]\times\mathbb{R})$ such that $v_y := \partial_y v$ is bounded. Then choosing measurable functions $\hat{\nu} : [0,T]\times\mathbb{R} \longrightarrow C$ and $\hat{\eta} : [0,T]\times\mathbb{R} \longrightarrow \mathbb{R}$ so that*

$$\begin{aligned} \hat{\nu}(t,y) &\in \arg\inf_{\nu\in C}\left(\frac{1}{2}(\nu_1 - \lambda(y))^2 + \nu_2\bar{\rho}v_y(t,y)\right) \\ \hat{\eta}(t,y) &= \hat{\nu}_2(t,y) - \bar{\rho}v_y(t,y), \end{aligned} \tag{2.7}$$

$(\hat{\nu}_\cdot, \hat{\eta}_\cdot) := (\hat{\nu}(\cdot, Y_\cdot), \hat{\eta}(\cdot, Y_\cdot))$ is an optimal control. In particular, $(Q^{\hat{\eta}}, P^{\hat{\nu}})$ is a solution to (1.2).

2.3. Optimal Strategy

We now give a representation of an optimal strategy $\hat{\theta}$ via Theorem 2.4 and the duality result of [5].

Theorem 2.5. *An optimal strategy $\hat{\theta} \in \Theta$ for the problem (1.1) is given by*

$$\hat{\theta}_t = \frac{\rho v_y(t,Y_t) + \lambda(Y_t) - \hat{\nu}_1(t,Y_t)}{\alpha\sigma(Y_t)S_t}. \tag{2.8}$$

Remark 2.6. Here we give a brief review of related literature. In the case without uncertainty i.e., $\mathcal{P} = \{R\}$ ($\Leftrightarrow C = \{(0,0)\}$ in our setup), explicit solutions to exponential hedging through duality are studied by [8] using BSDE arguments with the help of Malliavin calculus, and by [1] using PDE arguments close to ours.

There are also a few recent works deriving explicit form of optimal strategies for robust utility maximization. Our setup and idea for the proof of Theorem 2.4 are due to [3], where robust power utility maximization is considered. See also [4] for the case of logarithmic utility.

3. Explicit Examples

This section provides two explicit examples which are reduced to *linear* PDEs, hence can be computed either by an elementary numerical scheme or by the Feynman-Kac formula. Recall that our model is characterized by the compact set C, and the HJB equation takes the form:

$$\begin{cases} v_t + \mathcal{A}^0 v + \frac{\bar{\rho}^2 v_y^2}{2} - l(y, v_y), \\ v(T,y) = \alpha h(y), \end{cases}$$

where

$$l(y,p) := \inf_{\nu\in C}\left(\frac{1}{2}(\nu_1 - \lambda(y))^2 + \bar{\rho}\nu_2 p\right).$$

Thus, if $l(y,p)$ can be explicitly calculated, then we may expect an explicit solution.

3.1. The Case of Rectangle

Let C be a rectangle in $\mathbb{R}^2$, that is:

$$C = \{x \in \mathbb{R} : |x_1| \le m_1,\ |x_2| \le m_2\}. \tag{3.1}$$

In this case,

$$l(y,p) = \frac{1}{2}(\hat{\nu}_1(y) - \lambda(y))^2 + \bar{\rho}\hat{\nu}_2(p)p = \frac{k(y;m_1)}{2} - \bar{\rho}m_2|p|,$$

where

$$\hat{v}_1(y) = \mathrm{sgn}(\lambda(y))(|\lambda(y)| \wedge m_1), \quad \hat{v}_2(p) = -m_2 \mathrm{sgn}(p),$$
$$k(y; m_1) := \{(|\lambda(y)| - m_1)^+\}^2.$$

Therefore, the HJB equation is written as:

$$v_t + \mathcal{A}^0 v + \frac{\bar{\rho}^2 v_y^2}{2} + \bar{\rho} m_2 |v_y| - \frac{k(y; m_1)}{2} = 0. \tag{3.2}$$

Now suppose that the payoff function h is non-increasing. Then since the 1-dimensional stochastic flow associated to Y is order-preserving under (B1) and (B2), the value function is also non-increasing in y, hence $v_y \leq 0$. Therefore the term $\bar{\rho} m_2 |v_y|$ in (3.2) is replaced by $-\bar{\rho} m_2 v_y$. Moreover, *changing the drift*, the equation becomes:

$$v_t + \mathcal{A}^{\bar{\rho} m_2} v + \frac{\bar{\rho}^2 v_y^2}{2} - \frac{k(y; m_1)}{2} = 0.$$

Here $\mathcal{A}^{\bar{\rho} m_2}$ is the generator of Y under $Q^{\bar{\rho} m}$. This equation can be *linearized*. Note that

$$\begin{aligned} dv(t, Y_t) &= (v_t + \mathcal{A}^{\bar{\rho} m_2} v)(t, Y) dt + v_y(t, Y_t) d\bar{W}_t^{\bar{\rho} m_2} \\ &= \left(\frac{k(Y_t; m_1)}{2} - \frac{\bar{\rho}^2 v_y(t, Y_T)^2}{2} \right) dt + v_y(t, Y_t) d\bar{W}_t^{\bar{\rho} m_2}. \end{aligned}$$

For any γ,

$$\begin{aligned} de^{\gamma v(t, Y_t)} &= e^{\gamma v(t, Y_t)} \left[\gamma \left\{ \frac{k(Y_t; m_1)}{2} - \frac{\bar{\rho}^2 v_y(t, Y_T)^2}{2} \right\} + \frac{\gamma^2 v_y(t, Y_t)^2}{2} \right] dt \\ &\quad + e^{\gamma v(t, Y_t)} \gamma v_y(t, Y_t) d\bar{W}_t^{\bar{\rho} m_2} \end{aligned}$$

Setting $\gamma = \bar{\rho}^2$ and multiplying both sides by $e^{-\frac{1}{2}\int_0^t k(Y_s; m_1) ds}$, we have

$$de^{\bar{\rho}^2 v(t, Y_t) - \frac{1}{2}\int_0^t k(Y_s; m_1) ds} = e^{\bar{\rho}^2 v(t, Y_t) - \frac{1}{2}\int_0^t k(Y_s; m_1) ds} \bar{\rho}^2 v_y(t, Y_t) d\bar{W}_t^{\bar{\rho} m_2}.$$

Thus, $e^{\bar{\rho}^2 v(t, Y_t) - \frac{1}{2}\int_0^t k(Y_s; m_1) ds}$ is a martingale. Since $v(T, y) = \alpha h(y)$,

$$e^{\bar{\rho}^2 v(t, Y_t) - \frac{1}{2}\int_0^t k(Y_s; m_1) ds} = E^{\bar{\rho} m_2}[e^{\alpha \bar{\rho}^2 h(Y_T) - \frac{1}{2}\int_0^T k(Y_s; m_1) ds} | \mathcal{F}_t].$$

Rewriting this, we have

$$v(t, y) = \frac{1}{\bar{\rho}^2} \log \tilde{v}(t, y) := \frac{1}{\bar{\rho}^2} \log E^{\bar{\rho} m_2}[e^{\alpha \bar{\rho}^2 h(Y_T) - \frac{1}{2}\int_t^T k(Y_s; m_1) ds} | Y_t = y].$$

The Feynman-Kac formula yields:

Corollary 3.1. *Suppose that C is given by (3.1) and h is non-increasing. Then the value function v is given by $v(t, y) = \frac{1}{\bar{\rho}^2} \log \tilde{v}(t, y)$, where $\tilde{v}$ solves the linear Cauchy problem:*

$$\begin{cases} \tilde{v}_t + \mathcal{A}^{\bar{\rho} m_2} \tilde{v} - \frac{1}{2}\{(|\lambda(\cdot)| - m_1)^+\}^2 \tilde{v} \\ \tilde{v}(T, y) = e^{\alpha \bar{\rho}^2 h(y)}, \end{cases}$$

and $(\hat{\eta}, \hat{v}) = (m_2 - \bar{\rho}(\tilde{v}_y / \tilde{v})(\cdot, Y), \mathrm{sgn}(\lambda(Y))(|\lambda(Y)| \wedge m_1), m_2)$ is an optimal control. Finally, an optimal strategy for the robust exponential hedging is given by

$$\hat{\theta}_t = \frac{\rho}{\alpha \bar{\rho}^2} \frac{\tilde{v}_y(t, Y_t)}{\tilde{v}(t, Y_t) \sigma(Y_t) S_t} + \frac{\mathrm{sgn}(\lambda(Y_t))(|\lambda(Y_t)| - m_1)^+}{\alpha \sigma(Y_t) S_t}.$$

3.2. The Case of Disk

Next we consider the case where the set C is a disk in $\mathbb{R}^2$ with radius r:

$$C = \{x \in \mathbb{R}^2 : \|x\| \le r\}. \tag{3.3}$$

But due to a technical difficulty, we assume the drift b of S under R is identically zero, or equivalently, λ is identically zero. In this case,

$$\begin{aligned} l(y, p) &= \inf_{\|\nu\| \le r} \left(\frac{\nu_1^2}{2} + \bar{\rho}\nu_2 p \right) \\ &= -r\bar{\rho}|p|, \end{aligned}$$

and $\hat{\nu}(y, p) = (0, -r \cdot \mathrm{sgn}(p))$ is a minimizer. Then the HJB equation is written as:

$$v_t + \mathcal{A}^0 v + \frac{\bar{\rho}^2 v_y^2}{2} + r\bar{\rho}|v_y| = 0,$$

and the same argument as the previous subsection shows:

Corollary 3.2. *Suppose that C is given by (3.3), $\lambda \equiv 0$ and h is non-increasing. Then the value function is represented as*

$$v(t, y) = \frac{1}{\bar{\rho}^2} \log \tilde{v}(t, y),$$

where $\tilde{v}$ is the solution to the Cauchy problem:

$$\begin{cases} \tilde{v}_t + \mathcal{A}^{r\bar{\rho}}\tilde{v} = 0 \\ \tilde{v}(T, y) = e^{\alpha\bar{\rho}^2 h(y)}. \end{cases}$$

An optimal control is given by $(\hat{\eta}, \hat{\nu}) = (r - \bar{\rho}(\tilde{v}_y/\tilde{v})(\cdot, Y), 0, r)$. Finally, an optimal portfolio strategy is given by

$$\hat{\theta}_t = \frac{\rho}{\alpha\bar{\rho}^2} \frac{\tilde{v}_y(t, Y_t)}{\tilde{v}(t, Y_t)\sigma(Y_t)S_t}.$$

Remark 3.3. In both of these examples, the case with *non-decreasing* h can be treated in symmetric ways.

4. Proofs

Proof of Lemma 2.2. (A1) is guaranteed by [3, Lemma 3.1] and [7, Lemma 3.2]. Let $\lambda := b/\sigma$ which is bounded by (B1) and (B2). Therefore $dQ^0/dR := \mathcal{E}_T(-(\lambda(Y), 0) \cdot W)$ defines an equivalent local martingale measure. Since $R \in \mathcal{P}$ and $\mathcal{H}(Q^0|R) = E^R[\int_0^T \lambda(Y_s)^2 ds]/2 < \infty$, (A2) is satisfied. Also, (B3) implies that h is globally Lipschitz continuous, hence admits a constant K_h such that $|h(y)| \le K_h(1 + |y|)$ for all $y \in \mathbb{R}$. Now (A3) will be verified by checking that $\{e^{\gamma|h(Y_T)|}\mathcal{E}_T(-\nu \cdot W) : \nu \in \mathcal{I}_{\mathcal{P}}\}$ is bounded in $L^2(R)$ for any $\gamma > \alpha$. By the Cauchy-Schwarz inequality,

$$E^R\left[\left(e^{\gamma|h(Y_T)|}\mathcal{E}_T(-\nu \cdot W)\right)^2\right] \le E^R\left[e^{4\gamma|h(Y_T)|}\right]^{\frac{1}{2}} E^R\left[e^{-4\nu\cdot W_T}\right]^{\frac{1}{2}}. \tag{4.1}$$

Introducing another R-Brownian motion $\bar{W} = \rho W^1 + \bar{\rho} W^2$,

$$e^{4\gamma|h(Y_T)|} \le e^{4\gamma K_h(1+|Y_T|)} \le e^{4\gamma K_h(1+|Y_0|+\|g\|_\infty T+|\bar{W}_T|)}.$$

Hence a simple computation shows that the first component in the RHS of (4.1) is bounded by $\sqrt{2}e^{2\gamma K_h\{1+|Y_0|+(\|g\|_\infty+2\gamma K_h)T\}}$. For the second, we can apply [6, Th. III 39] to get an upper bound $e^{2T(\mathrm{diam}C)^2}$. Thus (A3) is verified, and the dual problem admits a maximal solution $(\widehat{Q}_H, \widehat{P}_H)$. Finally, (A4) is trivially satisfied since all $P \in \mathcal{P}$ are equivalent. □

For the proof of Theorem 2.4, we first consider a family of auxiliary control problems, restricting the domain of η. For each closed interval $I \subset \mathbb{R}$ (possibly $\mathbb{R}$ itself), set $\mathcal{I}_M^I := \{\eta \in \mathcal{I}_M : \eta_t \in I \ \forall t, \text{ a.s.}\}$, and consider the equation:

$$\partial_t v^I + \sup_{\eta \in I, \nu \in C} \left\{ \mathcal{A}^\eta v^I - \frac{1}{2}\|\nu - (\lambda(y), \eta)'\|^2 \right\} = 0, \quad v^I(T, y) = \alpha h(y). \tag{4.2}$$

If I is compact, then so is $I \times C$, hence we can apply Theorem 4.1 and 6.2 of Fleming and Rishel [2] to get:

Lemma 4.1. *For each compact $I \subset \mathbb{R}$, (4.2) admits a unique classical solution $v^I \in C_p^{1,2}((0,T) \times \mathbb{R}) \cap C([0,T] \times \mathbb{R})$. Then taking*

$$(\eta^I(t, y), \nu^I(t, y)) \in \arg \sup_{\eta \in I, \nu \in C} \left\{ \mathcal{A}^\eta v^I - \frac{1}{2}\|\nu - (\lambda(y), \eta)'\|^2 \right\},$$

we have

$$v^I(t, Y_t) = \operatorname*{ess\,sup}_{\eta \in \mathcal{I}_M^I, \nu \in \mathcal{I}_\mathcal{P}} J_t^{\eta,\nu} = J_t^{\eta^I(\cdot, Y), \nu^I(\cdot, Y)}.$$

Lemma 4.2. *There exists a constant K_v such that $|v_y^I| \le K_v$ for all compact I.*

Proof. Let $J_t^{\eta,\nu}(y) := E^\eta[\alpha h(Y_{t,T}(y)) - \frac{1}{2}\int_t^T \|\nu_s - (\lambda_s, \eta_s)'\|^2 ds]$, where $Y_{t,T}$ denotes the stochastic flow associated to Y. Then it suffices to show the existence of a constant K_v such that $|J_t^{\eta,\nu}(y) - J_t^{\eta,\nu}(y')| \le K_v |y - y'|$ for all $t \in [0,T]$, $y, y' \in \mathbb{R}$ and $(\eta, \nu) \in \mathcal{I}_M \times \mathcal{I}_\mathcal{P}$.

Since h and g are Lipschitz continuous with Lipschitz constants K_h, K_g,

$$\begin{aligned} |J_t^{\eta,\nu}(y) - J_t^{\eta,\nu}(y')| &= |E^\eta\left[\alpha h(Y_{t,T}(y)) - \alpha h(Y_{t,T}(y'))\right]| \\ &\le \alpha K_h E^\eta\left[|Y_{t,T}(y) - Y_{t,T}(y')|\right], \end{aligned}$$

and,

$$\begin{aligned} E^\eta\left[|Y_{t,T}(y) - Y_{t,T}(y')|\right] &\le |y - y'| + E^\eta\left[\int_t^T |g(Y_{t,s}(y) - g(Y_{t,s}(y'))| ds\right] \\ &\le |y - y'| + K_g \int_t^T E^\eta[|Y_{t,s}(y) - Y_{t,s}(y')|] ds \end{aligned}$$

Then the Gronwall inequality shows that $E^\eta[|Y_{t,T}(y) - Y_{t,T}(y')|] \le e^{K_g(T-t)}|y - y'| \le e^{K_g T}|y - y'|$. Hence $|J_t^{\eta,\nu}(y) - J_t^{\eta,\nu}(y')| \le \alpha K_h e^{K_g T}|y - y'|$. $\square$

Proof of Theorem 2.4. The inside of the bracket in (4.2) is written as:

$$\begin{aligned} \mathcal{A}^\eta v^I - \frac{1}{2}\|\nu - (\lambda(y), \eta)'\|^2 &= \mathcal{A}^0 v^I + \bar{\rho}(v_y^I)^2 - \frac{1}{2}\left\{\eta - (\nu_2 - \bar{\rho} v_y^I)\right\}^2 \\ &\quad - \left\{\frac{1}{2}(\lambda(y) - \nu_1)^2 + \nu_2 \bar{\rho} v_y^I\right\} \end{aligned}$$

Here the third term in the RHS attains the global maximum in η at $\eta^I = \nu_2 - \bar{\rho} v_y^I$, which is bounded by $\operatorname{diam}(C) + K_v$ independently of I. Therefore, taking $I_0 := [-\operatorname{diam}(C) - K_v, \operatorname{diam}(C) + K_v]$, we have

$$\begin{aligned} -\partial_t v^{I_0} &= \sup_{\eta \in I_0, \nu \in C} \left\{ \mathcal{A}^\eta v^{I_0} - \frac{1}{2}\|\nu - (\lambda(y), \eta)'\|^2 \right\} \\ &= \sup_{\eta \in \mathbb{R}, \nu \in C} \left\{ \mathcal{A}^\eta v^{I_0} - \frac{1}{2}\|\nu - (\lambda(y), \eta)'\|^2 \right\}. \end{aligned}$$

Hence $v := v^{I_0}$ is a desired classical solution to (2.6).

It remains to verify that $(\hat{\eta}, \hat{\nu})$ is an optimal control. By the Itô formula,

$$\begin{aligned}
v(t, Y_t) &= \alpha h(Y_T) - \int_t^T (v_t + \mathcal{A}^{\hat{\eta}} v)(s, Y_s) ds - \int_t^T v_y(s, Y_s) d\bar{W}_s^{\hat{\eta}} \\
&= \alpha h(Y_T) - \frac{1}{2}\int_t^T \|\hat{\nu}_s - (\lambda_s, \hat{\eta}_s)'\|^2 ds - \int_t^T v_y(s, Y_s) d\bar{W}_s^{\hat{\eta}} \\
&= E^{\hat{\eta}}\left[\alpha h(Y_T) - \frac{1}{2}\int_t^T \|\hat{\nu}_s - (\lambda_s, \hat{\eta}_s)'\|^2 ds | \mathcal{F}_t\right] \\
&= J_t^{\hat{\nu}, \hat{\eta}}.
\end{aligned}$$

Here $\lambda_s := \lambda(Y_s)$ and $\bar{W}^\eta := \rho W^{1,\eta} + \bar{\rho} W^{2,\eta}$. The second equality follows from (2.6) and the third from the $\mathcal{F}_t$-measurability of $v(t, Y_t)$ and boundedness of v_y. Also, for every $(\nu, \eta) \in \mathcal{I}_\mathcal{P} \times \mathcal{I}_M$,

$$\begin{aligned}
v(t, Y_t) &= \alpha h(Y_T) - \int_t^T (v_t + \mathcal{A}^{\eta_s} v)(s, Y_s) ds - \int_t^T v_y(s, Y_s) d\bar{W}_s^{\eta} \\
&\geq \alpha h(Y_T) - \frac{1}{2}\int_t^T \|\nu_s - (\lambda_s, \eta_s)'\|^2 ds - \int_t^T v_y(s, Y_s) d\bar{W}_s^{\eta} \\
&= J_t^{\eta, \nu}.
\end{aligned}$$

Thus we have $J_t^{\hat{\nu}, \hat{\eta}} = v(t, Y_t) \geq J_t^{\nu, \eta}$ a.s. for all t. This completes the proof. □

Proof of Theorem 2.5. By the duality, it suffices to show that $\hat{\theta} \in \Theta$ and

$$dQ^{\hat{\eta}}/dP^{\hat{\nu}} = e^{-\alpha(\hat{\theta}\cdot S_T - h(Y_T))}/E^{P^{\hat{\eta}}}[e^{-\alpha(\hat{\theta}\cdot S_T - h(Y_T))}].$$

Since v satisfies the HJB equation,

$$\begin{aligned}
\alpha h(Y_T) &= v(0, Y_0) + \int_0^T \left(\partial_t + \mathcal{A}^{\hat{\eta}}\right) v(s, Y_s) ds + \int_0^T v_y(s, Y_s) d\bar{W}_s^{\hat{\eta}} \\
&= v(0, Y_0) + \frac{1}{2}\int_0^T \|\hat{\eta}_s - \hat{\nu}_s\|^2 ds + \int_0^T v_y(s, Y_s) d\bar{W}_s^{\hat{\eta}} \\
&= v(0, Y_0) + \log\frac{dQ^{\hat{\eta}}}{dP^{\hat{\nu}}} + \int_0^T ((\lambda_s, \hat{\eta}_s) - \hat{\nu}_s) dW_s^{\hat{\eta}} + \int_0^T v_y(s, Y_s) d\bar{W}_s^{\hat{\eta}} \\
&= v(0, Y_0) + \log\frac{dQ^{\hat{\eta}}}{dP^{\hat{\nu}}} + \int_0^T (\rho v_y + \lambda - \hat{\nu}_1)(s, Y_s) dW_s^{1,\hat{\eta}}.
\end{aligned}$$

Rearranging the terms,

$$\begin{aligned}
\log\frac{dQ^{\hat{\eta}}}{dP^{\hat{\nu}}} &= -v(0, Y_0) + \alpha h(Y_T) - \int_0^T (\rho v_y + \lambda - \hat{\nu}_1)(s, Y_s) dW_s^{1,\hat{\eta}} \\
&= -v(0, Y_0) + \alpha h(Y_T) - \int_0^T \frac{(\rho v_y + \lambda - \hat{\nu}_1)(s, Y_s)}{\sigma(Y_s) S_s} \sigma(Y_s) S_s dW_s^{1,\hat{\eta}} \\
&= -v(0, Y_0) + \alpha(h(Y_T) - \hat{\theta} \cdot S_T).
\end{aligned}$$

Hence $dQ^{\hat{\eta}}/dP^{\hat{\nu}} = e^{v(0,Y_0)} e^{-\alpha(\hat{\theta}\cdot S_T - h(Y_T))}$. Finally,

$$\int_0^T \hat{\theta}_s^2 d\langle S\rangle_s = \frac{1}{\alpha^2}\int_0^T \{(\rho v_y + \lambda - \hat{\nu}_1)(s, Y_s)\}^2 ds$$

is bounded, hence $\hat{\theta} \cdot S$ is a martingale under every $Q \in \mathcal{Q}_f$. This concludes the proof. □

APPENDIX A. ON RELATIVE ENTROPY

This appendix gives a proof of the following fact appeared in Remark 2.3:

Proposition A.1. *Suppose $\mathcal{P}$ is defined by (2.2) and $Q \sim R$. Then the following two conditions are equivalent:*

(1) $$\mathcal{H}(Q|P) < \infty \quad \exists P \in \mathcal{P},$$

(2) $$\mathcal{H}(Q|P) < \infty \quad \forall P \in \mathcal{P}.$$

In particular, $\inf_{P\in\mathcal{P}} \mathcal{H}(Q|P) < \infty$ *if and only if* $\mathcal{H}(Q|R) < \infty$.

Lemma A.2. *Suppose that $\mathcal{P}$ is given by (2.2). Then for every pair $P, \bar{P} \in \mathcal{P}$,*

$$E^P\left[\left(\frac{d\bar{P}}{dP}\right)^2\right] < \infty. \tag{A.1}$$

PROOF. Let $dP/dR = \mathcal{E}(-\nu\cdot W)$ and $d\bar{P}/dR = \mathcal{E}(-\bar{\nu}\cdot W)$ with $\nu, \bar{\nu} \in \mathcal{I}_{\mathcal{P}}$. Note that there exists a constant K such that $\|\nu_t(\omega)\| \le K$ for all (t,ω), for all $\nu \in \mathcal{I}_{\mathcal{P}}$ since C is compact in the definition of $\mathcal{I}_{\mathcal{P}}$.

$$\begin{aligned}\frac{d\bar{P}}{dP} &= \frac{\mathcal{E}_T(-\bar{\nu}\cdot W)}{\mathcal{E}_T(-\nu\cdot W)}\\ &= \exp\left(-(\bar{\nu}-\nu)\cdot W_T - \frac{1}{2}\int_0^T (\|\bar{\nu}_s\|^2 - \|\nu_s\|^2)ds\right)\\ &= \exp\left(-(\bar{\nu}-\nu)\cdot W_T^P - \frac{1}{2}\int_0^T \|\bar{\nu}_s - \nu_s\|^2 ds\right),\end{aligned}$$

where $W^P = (W^{P,1}, W^{P,2})$ is a P-Brownian motion given by $W^{P,i} = W^i + \int_0^\cdot \nu^i ds$ $(i = 1,2)$. Set $M = -(\bar{\nu}-\nu)\cdot W^P$, which is a P-square integrable martingale with $\langle M\rangle_t = \int_0^t \|\bar{\nu}_s - \nu_s\|^2 ds \le 4K^2T$. Then we have

$$\left(\frac{d\bar{P}}{dP}\right)^2 = \mathcal{E}_T(2M_T)\cdot\exp\left(\int_0^T \|\bar{\nu}_s - \nu_s\|^2 ds\right).$$

Noting that $\mathcal{E}(2M)$ is a positive super martingale (actually a martingale by Novikov's criterion),

$$\begin{aligned}E^P\left[\left(\frac{d\bar{P}}{dP}\right)^2\right] &= E^P\left[\mathcal{E}_T(2M_T)\cdot\exp\left(\int_0^T \|\bar{\nu}_s - \nu_s\|^2 ds\right)\right]\\ &\le e^{4K^2T} E^P[\mathcal{E}_T(2M)]\\ &\le e^{4K^2T}.\end{aligned}$$

□

PROOF OF PROPOSITION A.1. The implication (2) ⇒ (1) is obvious. Observe that

$$\log\frac{dQ}{dP} = \log\frac{dQ}{d\bar{P}} + \log\frac{d\bar{P}}{dP} \le \log\frac{dQ}{d\bar{P}} + \log\left(\frac{d\bar{P}}{dP}\vee 1\right),$$

for all $P, \bar{P} \in \mathcal{P}$. Thus,

$$\mathcal{H}(Q|P) = E^Q\left[\log\frac{dQ}{dP}\right] \le \mathcal{H}(Q|\bar{P}) + E^Q\left[\log\left(\frac{d\bar{P}}{dP}\vee 1\right)\right],$$

Therefore, it suffices to show that if $\mathcal{H}(Q|\bar{P}) < \infty$, then $E^Q[\log(\frac{d\bar{P}}{dP} \vee 1)] < \infty$ for all $P \in \mathcal{P}$.

Note that the convex conjugate of the exponential function e^x is $y \log y - y$ ($y \geq 0$), i.e., $\sup_{x \in \mathbb{R}}(xy - e^x) = y \log y - y$. Hence, in particular

$$xy \leq y \log y - y + e^x.$$

Letting $x = \log(\frac{d\bar{P}}{dP} \vee 1)$ and $y = \frac{dQ}{d\bar{P}}$, we have

$$\begin{aligned}
&E^Q\left[\log\left(\frac{d\bar{P}}{dP} \vee 1\right)\right] \\
&\quad = E^{\bar{P}}\left[\frac{dQ}{d\bar{P}} \log\left(\frac{d\bar{P}}{dP} \vee 1\right)\right] \\
&\quad = E^{\bar{P}}\left[\frac{dQ}{d\bar{P}} \log \frac{dQ}{d\bar{P}}\right] - E^{\bar{P}}\left[\frac{dQ}{d\bar{P}}\right] + E^{\bar{P}}\left[\frac{d\bar{P}}{dP} \vee 1\right] \\
&\quad = \mathcal{H}(Q|\bar{P}) - 1 + E^P\left[\left(\frac{d\bar{P}}{dP}\right)^2 1_{\{d\bar{P}/dP > 1\}}\right] + \bar{P}\left(\frac{d\bar{P}}{dP} \leq 1\right) \\
&\quad \leq \mathcal{H}(Q|\bar{P}) + E^P\left[\left(\frac{d\bar{P}}{dP}\right)^2\right] < \infty,
\end{aligned}$$

by Lemma A.2. This proves (1) $\Rightarrow$ (2). □

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